

EXACT AND APPROXIMATE COMPRESSION OF TRANSFER
MATRICES FOR GRAPH HOMOMORPHISMS

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Abstract

The aim of this paper is to extend the previous work on transfer matrix compression in the case of graph homomorphisms. For H -homomorphisms of lattice-like graphs we demonstrate how the automorphisms of H , as well as those of the underlying lattice, can be used to reduce the size of the relevant transfer matrices. As applications of this method we give currently best known bounds for the number of 4- and 5-colourings of the square grid, and the number of 3- and 4-colourings of the three-dimensional cubic lattice. Finally, we also discuss approximate compression of transfer matrices.

1. *Introduction*

Transfer matrices are a standard tool in various branches of mathematics. In enumerative combinatorics they have been used for a long time to solve counting problems which can be described using graph homomorphisms. In statistical physics transfer matrices have been a well used tool for the computation of partition functions of various spin models. In ergodic theory transfer matrices are used to describe the behaviour of a class of dynamical system known as subshifts of finite type. Recently the many similarities between these uses have been put on a firm mathematical ground. That the calculation of entropies for \mathbb{Z}^d -subshifts of finite type are equivalent to counting graph homomorphisms from \mathbb{Z}^d into some graph H was demonstrated in [7]. In [6] it was proved that counting weighted graph homomorphisms is equivalent to computation of the partition function for statistical physics models satisfying a condition known as reflection positivity. The study of phase transitions in statistical physics models has also begun to be studied in the language of homomorphisms [3].

A general limitation for all these applications is that transfer matrices tend to grow fast with the size of the system considered, thus limiting the size of the system one can work with. A recent development in this area is the use of automorphisms of the underlying graphs to reduce the size of the matrices. This was done for a special case in [5] and was developed as a general method in [13]. Our aim here is first to show how this method, now called transfer matrix compression, in many cases can be taken even further than earlier applications, and also to discuss how one can make even greater size reductions if one is ready to settle for bounds of the computed entropies rather than exact values. For many applications the latter is sufficient.

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Let us put things on a firmer ground. A homomorphism ϕ from a graph G to a graph H , which may have loops, is a mapping which preserves adjacencies, i.e. if $(x, y) \in E(G)$ then $(\phi(x), \phi(y)) \in E(H)$. The set of all homomorphisms from G to H is denoted $\text{Hom}(G, H)$. We say that H is a weighted graph if there are two functions $\alpha_H : V(H) \rightarrow F$ and $\beta_H : E(H) \rightarrow F$, where F is a ring. Given a weighted graph H we assign a weight $w(\phi)$ to each homomorphism ϕ from G to H

$$w(\phi) = \prod_{x \in V(G)} \alpha_H(\phi(x)) \prod_{xy \in E(G)} \beta_H(\phi(x), \phi(y)). \quad (1)$$

Let us give a few examples:

EXAMPLE 1.1.

1. $H = K_q$, $\alpha = \beta = 1$, corresponds to ordinary proper q -colourings.
2. Let H be a K_2 with a loop on one vertex, let the edge have weight 1, the loop weight t and $\alpha = 1$. For $t = 1$ the homomorphisms correspond to independent sets and for general t , we have the so-called hard-core lattice gas model.
3. If H is a K_q with loops on every vertex and the weights are t^{-1} on the loops, t on the ordinary edges and 1 on the vertices, we have the q -state Potts model.
4. If we take the previous example with $q = 2$ and put a weight s on one vertex of H and s^{-1} on the other vertex, we have the Ising model with an external field.

If we have a model where we also want to put colours or ‘spins’ on the edges of G (for example, when considering matchings, or on both edges and vertices), we can instead consider the line-graph or total-graph of the underlying graph G respectively. It is also straightforward to generalise these concepts to hypergraphs if one wants to consider interaction between larger sets of vertices.

We next define a weighted counter for these homomorphisms

$$Z(G, H) = \sum_{\phi \in \text{Hom}(G, H)} w(\phi).$$

If all weights on H are just 1 this will be exactly $|\text{Hom}(G, H)|$. In spin models the weights are often taken to be of the form e^K for a parameter K which is interpreted as a temperature, and then $Z(G, H)$ is called the partition function of the model. In ergodic theory $Z(G, H)$ is called the pressure of the subshift described by G and H . In most applications the aim is either to compute $Z(G, H)$ when F is a ring of polynomials, as in [9], or to determine how fast $Z(G_n, H)$ grows when $F = \mathbb{R}$ and G_n is some sequence of graphs; see, for example, [7, 5].

2. Polygraphs and transfer matrices

Transfer matrices are most useful for computation within a class of graphs known as polygraphs. This class was introduced in [1], where the transfer matrices were used to compute matching polynomials. A polygraph \mathcal{G} is defined by a set of disjoint graphs G_1, G_2, \dots, G_m and a set of binary relations $\Upsilon_1, \Upsilon_2, \dots, \Upsilon_m$, where $\Upsilon_i \subset V(G_i) \times V(G_{i+1})$. The vertex set of \mathcal{G} is $\bigcup_i V(G_i)$ and the edge set is

$$E(\mathcal{G}) = \bigcup_i E(G_i) \bigcup_i \Upsilon_i.$$

If all $G_i = G$ and $\Upsilon_i = \Upsilon$ for all $i = 1, \dots, m$ we write the corresponding polygraph as $\mathcal{G}(G, \Upsilon, m)$.

Given a polygraph \mathcal{G} and a weighted graph H we can compute $Z(\mathcal{G}, H)$ using a sequence of transfer matrices. We define a matrix $M(i)$ for going from G_i to G_{i+1} as follows.

Let $\Phi(G_i)$ denote the set of restrictions of all homomorphisms in $\text{Hom}(\mathcal{G}, H)$ to G_i . We call a member of $\Phi(G_i)$ a *state* on G_i . Now let the rows of $M(i)$ be indexed by the states on G_i and the columns by the states on G_{i+1} . We set $M_{x,y}(i) = 0$ if there is no homomorphism $\phi \in \text{Hom}(\mathcal{G}, H)$ such that $\phi|_{G_i} = x$ and $\phi|_{G_{i+1}} = y$. If there exists such a ϕ we set $M_{x,y}$ equal to the contribution to the weight $w(\phi)$ of the edges in Υ_i and the edges and vertices of G_{i+1} . We also define an associated vector η . The position in η corresponding to the row x is set equal to the weight of the partial homomorphism x .

The partition function is now given by

$$Z(\mathcal{G}, H) = \eta \left(\prod_i M(i) \right) \mathbf{1}. \tag{2}$$

One can also consider cyclic polygraphs where the last relation Υ connects G_m to G_1 . In this case the partition function is given by the trace of the transfer matrix product,

$$Z(\mathcal{G}, H) = \text{Tr} \left(\prod_i M(i) \right). \tag{3}$$

3. Exact compression of transfer matrices

Henceforth we will assume that our polygraphs are on the form $\mathcal{G}(G, \Upsilon, m)$. Most of what follows can be adapted to general polygraphs as well. Let us recall that given an $N \times N$ matrix M a partition $\mathcal{X} = \{X_1, X_2, \dots, X_r\}$ of $\{1, \dots, N\}$ is called an *equitable partition* if $\sum_{l \in X_j} M(i_1, l) = \sum_{l \in X_j} M(i_2, l)$ when $i_1, i_2 \in X_i$.

Given a partition X of the states on G , we define the compressed transfer matrix for $Z(\mathcal{G}, H)$ to be

$$C_{\mathcal{X}}(i, j) = \sum_{l \in X_j} M(k, l), \quad k \in X_i, \quad i, j = 1 \dots r. \tag{4}$$

The main theorem of [13] can be stated as follows.

THEOREM 3.1. *If X is an equitable partition of M , then*

$$C_{\mathcal{X}}^n(i, j) = \sum_{l \in X_j} M^n(k, l), \quad k \in X_i, \quad i, j = 1 \dots r \tag{5}$$

This has the following corollary.

COROLLARY 3.2. *Let η be the vector of length N whose i th entry is $|X_i| w(x_i)$, where $x_i \in X_i$. Then*

$$Z(\mathcal{G}, H) = \eta \left(\prod_i M(i) \right) \mathbf{1}.$$

The main consequence of these results, which has been used in [13, 5, 7], and [15, 10, 11, 4] is as follows.

COROLLARY 3.3. *If M is a transfer matrix and the partition \mathcal{X} consists of orbits on the set of states on G under the automorphism group $\text{Aut}(G)$, then \mathcal{X} is equitable.*

This corollary lets us make use of automorphisms of G to compress our transfer matrices and for graphs with reasonably large automorphism group, such as cycles, the reduction in size can be substantial.

EXAMPLE 3.4. Let us look at the transfer matrix for $\text{Hom}(G(P_3, \text{Id}, n), K_3)$, that is, 3-colourings of the graph $P_3 \times P_n$.

There are twelve states on P_3 and the only nontrivial member of $\text{Aut}(P_3)$ is a reflection in the midpoint. If we use 1,2,3 to denote colours, we find that there are nine orbits:

$$\{\{121\}, \{212\}, \{313\}, \{131\}, \{232\}, \{323\}, \{123, 321\}, \{132, 231\}, \{213, 312\}\}.$$

Here we get the following 9×9 matrix, instead of a 12×12 matrix,

$$C = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad \eta = \{1, 1, 1, 1, 1, 1, 2, 2, 2\}.$$

Even though we have reduced the side of the matrix by one quarter, we still have a fairly sparse matrix. The fact that we still have a sparse compressed matrix can on one hand be viewed a sign that the matrix could be compressed further to a dense matrix, as we will soon do, but for some cases it could also be taken advantage of by instead changing to computational techniques optimized for sparse matrices.

From algebraic graph theory [8] we know that if G is any graph and X is a partition of its vertices into orbits under $\text{Aut}(G)$ then the corresponding partition is equitable. If we choose to interpret the transfer matrix M as the adjacency matrix of a weighted graph, then what we did in Corollary 3.3 can be interpreted as using the subgroup of $\text{Aut}(M)$ which is induced by $\text{Aut}(G)$ to partition M . However, $\text{Aut}(M)$ has an even larger subgroup induced by $\text{Aut}(G) \times \text{Aut}(H)$. Here $\text{Aut}(H)$ is assumed to preserve the weights on H as well as the adjacencies.

COROLLARY 3.5. *Let X be a partition of M given by the orbits of $\text{Aut}(G) \times \text{Aut}(H)$; then X is equitable.*

When the graph H is highly symmetric, such as in the case of proper colourings of G or the partition function of the Potts model, the extra reduction in size achieved here can be remarkable.

EXAMPLE 3.6. If we use the automorphism group of K_3 as well in Example 3.4, we now only find two orbits,

$$\{\{121, 212, 313, 131, 232, 323\}, \{123, 321, 132, 231, 213, 312\}\},$$

and the compressed transfer matrix has side 2:

$$C = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}, \quad v = \{6, 6\}.$$

4. Application to the asymptotic number of q -colourings of lattices

The number of proper q -colourings of an $n \times n$ square grid, or $P_n \times P_n$, is known to grow exponentially as a function of n^2 . We denote the basis for this exponential growth by $\lambda^s(q)$. This is a quantity which is of interest both in enumerative combinatorics and statistical physics. In the latter case, $\log \lambda^s(q)$ is seen as the ground state entropy of the much studied antiferromagnetic Potts model on the square grid; see, for example, [16] for a survey.

For 3-colourings Lieb [12] found the exact asymptotic value of $\lambda^s(3) = (\frac{4}{3})^{3/2}$ but for larger q the value of $\lambda^s(q)$ is still unknown. More recently the antiferromagnetic Potts model has been studied, using transfer matrix and compression thereof, both in a series of papers by Jacobsen, Salas and Sokal [15, 10, 11], as well as in a collection of papers by Chang and Shrock; see, for example, [4].

As a full-scale example of our methods we now look at the compression of transfer matrices for q -colourings of the square grid for $q = 4, 5$, for the cubic grid $P_n \times P_n \times P_n$ for $q = 3, 4$, and their use in getting bounds for $\lambda^s(q)$.

It is known that the maximum eigenvalues $\theta_1(k)$ and $\theta_2(k)$ of the transfer matrices for $\text{Hom}(G(P_k, \text{Id}, n), K_q)$ and $\text{Hom}(G(C_k, \text{Id}, n), K_q)$ respectively, can be used to give upper and lower bounds for $\lambda^s(q)$; see, for example, [7] for a general treatment. In particular

$$\frac{\theta_1(k+1)}{\theta_1(k)} \leq \lambda^s(q) \leq \theta_2(2k)^{1/2k}. \quad (6)$$

So we can bound $\lambda^s(q)$ by computing $\theta_1(k)$ for consecutive k and $\theta_2(k)$ for even k .

In Table 1 we have given the size of the transfer matrix for $\text{Hom}(G(P_k, \text{Id}, n), K_3)$ and $\text{Hom}(G(C_k, \text{Id}, n), K_3)$. Here N_1 denotes the side of the uncompressed transfer matrix, N_2 the side when the automorphism groups of P_k and C_k respectively were used, and N_3 the size when the automorphism group of K_3 was used as well. The effect of the larger automorphism group of the cycle is easily visible, as is the gain from including the automorphism group of K_4 in the compression step. We can find N_1 exactly by evaluating the chromatic polynomial; see, for example, [14], of the cycle and the path, giving us $N_1 = 3^k + (-1)^k 3$ and $N_1 = 43^{k-1}$ for the cycle and paths respectively. For both the path and the cycle, N_1 grows with $\mathcal{O}(3^k)$. A typical colouring of, for example, the cycle is not invariant under any non-trivial automorphisms so we expect N_2 to be $\mathcal{O}(3^n / |\text{Aut}(C_k)|) = \mathcal{O}(3^k / 2k)$ and N_3 to be almost a factor 4! smaller, and for larger k this is indeed close to what we see in Table 1.

Here we can also note an additional property that will hold for larger values of q , as well as for homomorphisms to other highly symmetrical graphs H . A q -colouring of a graph G is equivalent to a partition of the vertex set of G into independent sets together with an assignment of distinct colours to the parts of this partition. When an element $\sigma \in \text{Aut}(G) \times \text{Aut}(H)$ acts on a colouring, or homomorphism in general, the part of σ from $\text{Aut}(G)$ acts on the parts of the partition by moving

them on the graph, and the part of σ from $\text{Aut}(H)$ acts by changing the labels of the parts. When q is larger than the order of G , the number of parts in a partition of $V(G)$ must be smaller than q , and as q increases N_1 can only increase by increasing the number of possible labelings of the parts of the partitions. This means that while N_1 will continue to grow with q , for a fixed G , the value of N_3 will not. This observation has been used in [15], where q -colourings for large values of q were studied. Using this observation, one can also bypass the step of computing the full set of colourings corresponding to N_1 and instead start out with partitions of the vertex set of G , but in the current paper we will stay with the basic method of starting out with all colourings of G .

For small k , the transfer matrices and eigenvalues were computed first with a MATHEMATICA program and also with a FORTRAN 90 program. For larger k the FORTRAN 90 program was run on a LINUX cluster. In Table 2 of Appendix A we have collected the computed eigenvalues. The higher-precision values for small k are due to the MATHEMATICA program.

Using these eigenvalues and inequalities (6) we find the following bounds for $\lambda^s(4)$:

$$2.336056640723116 \leq \lambda^s(4) \leq 2.33606820555777. \tag{7}$$

To our knowledge these are currently the best rigorous bounds for $\lambda^s(4)$. In [2] the first terms of a series expansion in $1/(q - 1)$ for $\lambda^s(q)$ was obtained, and using this series it was estimated that

$$\lambda^s(4) = 2.336056641 \pm 0.000\ 000\ 001,$$

with a heuristic error bound, an estimate which fits in just above our lower bound.

In the same way we computed the corresponding eigenvalues for 5-colourings, given in Table 3 of Appendix A. The bounds so obtained for $\lambda^s(5)$ are

$$3.2504049231640764 \leq \lambda^s(5) \leq 3.250407145038276. \tag{8}$$

For the growth rate $\lambda^c(q)$ of the number q -colourings of the cubic lattice $P_k \times P_k \times P_k$ there are no exact results known. In [2] series estimates were also given for $\lambda^c(3)$ and $\lambda^c(4)$, however these estimates were based on much shorter series than

Table 1: Transfer matrix sizes for 4-colourings. The upper three rows are for transferring a path, and the lower three for a cycle.

n	3	4	5	6	7	8	9	10	11
N_1	36	108	324	972	2916	8748	26244	78732	236196
N_2	24	54	180	486	1512	4374	13284	39366	118584
N_3	2	4	10	25	70	196	574	1681	5002
N_1	24	84	240	732	2184	6564	19680	59052	177144
N_2	4	21	24	92	156	498	1096	3210	8052
N_3	1	3	2	9	10	34	57	169	366

those for the square grid and were given as

$$\begin{aligned}\lambda^c(3) &= 1.4435 \pm 0.0005; \\ \lambda^c(4) &= 2.043 \pm 0.001.\end{aligned}$$

In order to compute bounds for the cubic lattice we can make use of the observations that $\lambda^c(q)$ is greater than the maximum eigenvalue for $\text{Hom}(G(P_k \times C_\ell, \text{Id}, n), K_q)$, since colourings of these graphs can be extended to periodic colourings of $\text{Hom}(G(P_k \times P_\ell^t, \text{Id}, n), K_q)$ for all t . Likewise, $\lambda^c(q)$ is less than the maximum eigenvalue for $\text{Hom}(G(P_k \times P_\ell, \text{Id}, n), K_q)$, since the number of colourings is submultiplicative.

As before, we can get lower bounds for the maximum eigenvalue of $\text{Hom}(G(P_k \times C_\ell, \text{Id}, n), K_q)$ for a fixed ℓ by computing the maximum eigenvalues for consecutive k , and for each value of ℓ we will get a lower bound for $\lambda^c(q)$. Similarly we can get upper bounds for the maximum eigenvalue of $\text{Hom}(G(P_k \times P_\ell, \text{Id}, n), K_q)$ by computing the maximum eigenvalue of $\text{Hom}(G(C_k \times P_\ell, \text{Id}, n), K_3)$ for even k . These eigenvalues are in turn bounded from above by the maximum eigenvalue of $\text{Hom}(G(C_k \times C_\ell, \text{Id}, n), K_3)$, for even k and ℓ .

For $\lambda^c(3)$ our best bounds come from the eigenvalues of $\text{Hom}(G(C_6 \times C_6, \text{Id}, n), K_q)$ and $\text{Hom}(G(P_k \times C_4, \text{Id}, n), K_q)$; for $\lambda^c(4)$ the bounds were achieved by $\text{Hom}(G(C_4 \times C_4, \text{Id}, n), K_q)$ and $\text{Hom}(G(P_k \times C_4, \text{Id}, n), K_q)$:

$$1.4460096817417 \leq \lambda^c(3) \leq 1.4470681274660; \tag{9}$$

$$2.0343787307189 \leq \lambda^c(4) \leq 2.0652128520667. \tag{10}$$

As we can see, the estimate from [2] for $\lambda^c(4)$ is within our bounds but their estimate for $\lambda^c(3)$ is well below the lower bound, even when their heuristic error estimate is taken into account. As mentioned in [2] this kind of ‘miss’ by the series estimate could indicate a physically interesting structure in the set of 3-colourings.

5. Approximate compression

For many applications, the ring F is the real numbers and one typically has only positive weights. If the aim is to compute only the maximum eigenvalue of M , as in Example 4, we can go further with our compression than in the previous section. First we here only need to care about the so-called ‘main part’ of the spectrum of M , that is, the eigenvalues with eigenvector not orthogonal to $\mathbf{1}$, and of the main part we need only to preserve the maximum eigenvalue. We can now make use of one of the standard theorems of spectral graph theory; see, for example, [8].

THEOREM 5.1 (Interlacing of eigenvalues). *Let S be an $n \times m$ matrix such that $S^T S = I$ and M a hermitian $n \times n$ matrix, and set $M' = S^T M S$. Let the eigenvalues of M be $\lambda_1 \geq \lambda_2, \dots, \lambda_n$ and those of M' be $\theta_1 \geq \theta_2, \dots, \geq \theta_m$. Then the eigenvalues of M' interlace the eigenvalues of M ; that is,*

$$\lambda_j \geq \theta_j \geq \lambda_{n-m+j}.$$

COROLLARY 5.2. *Let M be an $m \times m$ hermitian matrix and let $X = \{X_1, X_2, \dots\}$ be a partition of $\{1, \dots, m\}$. Define a matrix B where $B_{i,j}$ is the average row sum in M_{X_i, X_j} . Then the eigenvalues of B interlace those of M .*

Proof. Apply Theorem 5.1 with the matrix S given by $S_{i,j} = |X_i|^{-1/2}$ if $j \in X_i$ and 0 elsewhere. \square

For a partition which is not necessarily equitable we thus find the following corollary.

COROLLARY 5.3. *Let \mathcal{X} be a partition of the rows and columns of M ; then the maximum eigenvalue of $C(\mathcal{X})$ gives a lower bound on $\lambda_1(M)$.*

Given a partition \mathcal{X} , we can also define a matrix D , defined as in the previous corollary but using the maximum row sum rather than the average.

COROLLARY 5.4. *The maximum eigenvalue of D gives an upper bound on $\lambda_1(M)$.*

For many choices of weighted graph H it is the case that $Z(G, H)$ is either sub- or super-additive with respect to the addition of edges and/or vertices to G . In this situation the corollaries of the interlacing theorem can be used to give us upper and lower bounds on the asymptotics of the maximum eigenvalues as G becomes larger. These bounds can then in turn be used in combination with inequalities like (6).

Apart from getting bounds on the eigenvalues of M , one can also use an approximately compressed transfer matrix to speed up iterative algorithms for computing the exact maximum eigenvalue of M . Many such algorithms, such as the basic Perron-iteration, start out with a vector v_0 , and the number of iterations in the algorithms depends on how close v_0 is to the eigenvector of the maximum eigenvalue. Using an approximately compressed transfer matrix, we can quickly find the maximum eigenvector of the compressed matrix; next this can vector can be expanded to a vector v_0 by replicating the i th entry in the shorter vector in each of the $|X_i|$ entries corresponding to that entry in the uncompressed vector.

In both Corollary 5.3 and Corollary 5.4 the choice of partition \mathcal{X} will influence the value of the eigenvalue bound. How the partition should be chosen in order to get a good approximation will depend on the underlying graphs and weights, and it is hard to say anything much more precise than that one should strive to get blocks M_{X_i, X_j} with as closely concentrated row-sums as possible.

EXAMPLE 5.5. In order to demonstrate the approximate bounds, and the influence of the choice of partition \mathcal{X} , we have computed these bounds for the transfer matrix for the number of 4-colourings on $P_{12} \times P_n$ and $C_{14} \times P_n$.

We have used two kinds of partition \mathcal{X} .

1. For the first type of partition we viewed each colouring as an integer written in base 4; for cycles we chose an arbitrary vertex to be the lowest digit. Next we sorted the colourings as if they were integers.

Given this sorted list of the colourings, we partitioned the list into consecutive sublists of length k and $k - 1$, with as few lists of length $k - 1$ as possible.

2. As our second kind of partition we randomly partitioned the list of colours into parts of size K .

For these two partitions we next computed an upper bound for the largest eigenvalue of the transfer matrix for C_{14} and a lower bound for that of P_{12} . For each graph and partitioning we chose K so as to give compressed matrices with sides from 0.9 times the original side down to 0.1. For the random ordering, we tried several random partitions.

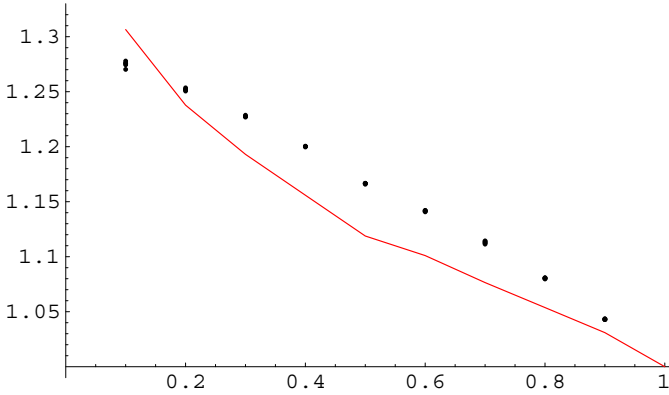


Figure 1: Approximate eigenvalues for C_{14} . Connected points are from the integer encoding. Clusters of isolated points are random partitions.

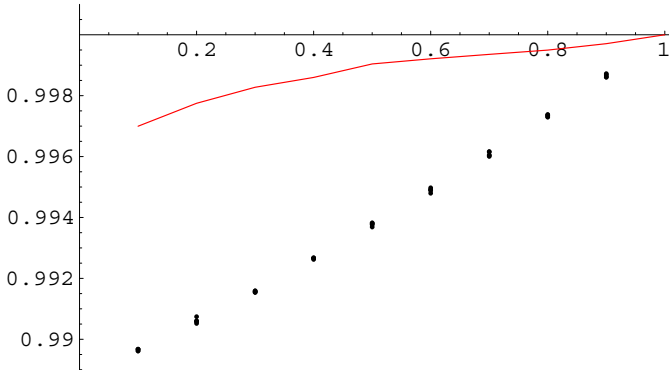


Figure 2: Approximate eigenvalues for P_{12} . Connected points are from the integer encoding. Clusters of isolated points are random partitions.

In Figures 1 and 2 we have plotted the approximate eigenvalue divided by the correct eigenvalue. We find that the lower bounds tend to be more accurate than the upper bounds. In both cases the integer encoding partition gives a noticeably better approximation than the random partitions. However, for the lower bound even the random partitions can be used to compress the matrix down to one tenth of its original size and still get a bound which is just one percent less than the correct value.

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Appendix A. *Colouring eigenvalues for the square grid*

Table 2: Maximum eigenvalues for 4-colourings of the square grid

k	P_k	C_k
3	16.34846922834953429459185	11
4	38.18874899819785577572648	31.69693845669906858918370
5	89.20972864650976523895547	67.01514803843835560759098
6	208.3980964253975720633538	165.6008220944556672481883
7	486.8291555413566000543864	375.4804004152886797996016
8	1137.260058429224259797968	892.2418753486577354212783
9	2656.703606588566986303095	2064.554606528212648447034
10	6206.209878666071640432711	4849.504943339923099642784
11	14498.05763470071954268293	11293.10916510243643781710
12	33868.2836924334	26429.64958444568607749808
13	79118.2289428612	61675.61597454731
14	184824.664073982	144167.2612085567
15	431760.883879445	336660.4085235824
16		786626.0015010700

Table 3: Maximum eigenvalues for 5-colourings of the square grid

n	P_n	C_n
2	13	
3	42.254746265138	32
4	137.34484848076	114.16796064692
5	446.42629917197	359.90932515034
6	1451.0662111085	1182.6934618883
7	4716.5527439510	3829.1466667249
8	15330.706253905	12464.383871815
9	49831.003080926	40492.334305935
10	161970.93773951	131643.39169572
11	526471.13343803	427861.13442804
12		1390763.1270219

Appendix B. *Colouring eigenvalues for the cubic lattice*

Table 4: Maximum eigenvalues for 3-colourings of $G \times P_2$

n	$G = P_n$	$G = C_n$
3	13.8072589673052	3
4	30.1109468160278	26.6214214843774
5	65.8601729387350	24.6080079665097
6	144.283083091960	123.199451464237
7	316.392676802175	148.219336059660
8	694.239161184508	582.950876259572
9	1523.97594483322	806.285660630712
10	3346.41981099416	2782.03759223168
11	7349.89922843146	4196.30975248900

Table 5: Maximum eigenvalues for 3-colourings of $G \times P_3$

n	$G = P_n$	$G = C_n$
3	42.9509955498558	4.56155281280883
4	134.633390548866	114.548378741056
4	423.398960388624	103.444398290072
5	1333.80481197401	1091.43690942498
6	4206.08745616625	1449.29878537714
7		10674.0945361673

Table 6: Maximum eigenvalues for 3-colourings of $G \times P_4$

n	$G = P_n$	$G = C_n$
4	607.5008342289296	496.9033949197111
5	2751.292994653581	437.9397858090114
6	12483.36568754961	9768.207310946096

Table 7: Maximum eigenvalues for 3-colourings of $G \times P_5$

n	$G = P_n$	$G = C_n$
5	17953.38896417563	1859.891162040439'

Table 8: Maximum eigenvalues for 3-colourings of $G \times C_3$

n	$G = P_n$	$G = C_n$
2	3	
3	4.56155281280883	2
4	6.97196076839709	6.37228132326901
5	10.6828851212084	6
6	16.3920411989578	14.5064314940480
7	25.1740785316175	15.7833418763922
8	38.6831608665319	33.6767869577220
9	59.4651079147947	39.6505660120334
10	91.4379622705829	78.8188645918277
11	140.631735559805	97.5298788390751
12	216.327079158290	185.239857806635
13	332.808012753772	237.182998032027

Table 9: Maximum eigenvalues for 3-colourings of $G \times C_4$

n	$G = P_n$	$G = C_n$
2	26.62142148437744	
3	114.5483787410569	
4	496.9033949197111	420.477039628259
5	2163.237391033718	378.843114768611
6	9435.406059898469	7704.08921920854
7		11291.7201866529
8		144633.687249454

Table 10: Maximum eigenvalues for 3-colourings of $G \times C_5$

n	$G = P_n$	$G = C_n$
2	24.6080079665097	
3	103.444398290072	
4	437.939785809011	
5	1859.89116204043	408.155175807023
6	7912.06577168573	6610.84386549256

Table 11: Maximum eigenvalues for 3-colourings of $G \times C_6$

n	$G = P_n$	$G = C_n$
6		599243.330687515

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