

# ON CONJUGACY CLASSES OF CONGRUENCE SUBGROUPS OF $\text{PSL}(2, \mathbb{R})$ .

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## *Abstract*

Let  $G$  be a subgroup of  $\text{PSL}(2, \mathbb{R})$  which is commensurable with  $\text{PSL}(2, \mathbb{Z})$ . We say that  $G$  is a *congruence subgroup* of  $\text{PSL}(2, \mathbb{R})$  if  $G$  contains a principal congruence subgroup  $\bar{\Gamma}(N)$  for some  $N$ . An algorithm is given for determining whether two congruence subgroups are conjugate in  $\text{PSL}(2, \mathbb{R})$ . This algorithm is used to determine the  $\text{PSL}(2, \mathbb{R})$  conjugacy classes of congruence subgroups of genus-zero and genus-one. The results are given in a table.

## 1. *Introduction.*

The principal congruence subgroups of  $\Gamma = \text{SL}(2, \mathbb{Z})$  are defined as follows:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\},$$

for  $N = 1, 2, 3, \dots$ . The image of  $\Gamma(N)$  in  $\bar{\Gamma} = \text{PSL}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})/\{\pm 1_2\}$  is denoted by  $\bar{\Gamma}(N)$ . A subgroup of  $\bar{\Gamma}$  is called a congruence subgroup if it contains some  $\bar{\Gamma}(N)$ . The level of a congruence subgroup  $\bar{G}$  of  $\bar{\Gamma}$  is the smallest  $N$  such that  $\bar{\Gamma}(N)$  is contained in  $\bar{G}$ .

It is natural to extend this definition to subgroups of  $\text{PSL}(2, \mathbb{R})$ . A subgroup of  $\text{PSL}(2, \mathbb{R})$  is commensurable with  $\bar{\Gamma}$  if  $\bar{G} \cap \bar{\Gamma}$  has finite index in both  $\bar{G}$  and  $\bar{\Gamma}$ . In this case, we say that  $\bar{G}$  is a congruence subgroup if it contains some  $\bar{\Gamma}(N)$ .

It was originally conjectured by Rademacher that there are only finitely many congruence subgroups of given genus in  $\bar{\Gamma}$ . This problem was studied by several authors. Cox and Parry [3, 4] gave effective bounds and computed a list of genus-zero congruence subgroups of  $\bar{\Gamma}$ . Independently, Thompson [9] showed that, up to conjugation, there are only finitely many congruence subgroups of  $\text{PSL}(2, \mathbb{R})$  of fixed genus. Motivated by Thompson’s result, and using bounds due to Zograf [10], a list of congruence subgroups of  $\text{PSL}(2, \mathbb{R})$  of genus-zero and genus-one was found in [5] and fundamental domains for these groups were found in [6].

The strategy of [5] was, following Thompson, to use a result of Helling which states that every group commensurable with  $\bar{\Gamma}$  is conjugate to a subgroup of a certain class of maximal discrete subgroups of  $\text{PSL}(2, \mathbb{R})$ . Thus to find all congruence subgroups of genus-zero and genus-one, it suffices to compute all the conjugacy classes of such subgroups inside these “Helling Groups”. Further work, however, is required to find the resulting  $\text{PSL}(2, \mathbb{R})$  conjugacy classes. In this paper we supply the necessary algorithm and the resulting conjugacy classes are given in Table 1.

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In Section 2 background definitions are given. The algorithm for the computation is described in Section 3 and some concluding comments are given in Section 4.

## 2. Background

If  $\overline{G}$  is a discrete subgroup of  $\mathrm{PSL}(2, \mathbb{R})$  which is commensurable with  $\overline{\Gamma}$ , then  $\overline{G}$  acts on the extended upper half plane  $\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$  by fractional linear transformations and the genus of  $\overline{G}$  is defined to be the genus of the corresponding Riemann surface  $\mathcal{H}^*/\overline{G}$ .

From a computational point of view, it is easier to work with subgroups of  $\Gamma$  and  $\mathrm{SL}(2, \mathbb{R})$ , rather than  $\overline{\Gamma}$  and  $\mathrm{PSL}(2, \mathbb{R})$ . There is a 1-1 correspondence between the subgroups of  $\mathrm{PSL}(2, \mathbb{R})$  and the subgroups of  $\mathrm{SL}(2, \mathbb{R})$  which contain  $-1_2$ , where  $1_2$  is the identity of  $\mathrm{SL}(2, \mathbb{R})$ . Thus in [5] and in this paper we consider subgroups of  $\mathrm{SL}(2, \mathbb{R})$  which contain  $-1_2$ . If  $G$  is subgroup of  $\mathrm{SL}(2, \mathbb{R})$  and  $\overline{G}$  is its image in  $\mathrm{PSL}(2, \mathbb{R})$ , then when we refer to geometric invariants such as the genus or cusp number of  $G$ , we mean the corresponding invariants of  $\overline{G}$ .

We recall the following definition.

DEFINITION 2.1.

$$\Gamma_0(f)^+ = \left\{ e^{-1/2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}) \mid a, b, c, d, e \in \mathbb{Z}, e \parallel f, e \mid a, e \mid d, \right. \\ \left. f \mid c, ad - bc = e \right\},$$

where  $e \parallel f$  means  $e \mid f$  and  $\mathrm{gcd}(e, f/e) = 1$ .

By the following theorem, the study of groups commensurable with  $\Gamma$  is essentially the study of subgroups of the groups  $\Gamma_0(f)^+$ , where  $f$  is a square-free integer.

THEOREM 2.2. (Helling [7]. See also Conway [2]). *If  $G$  is a subgroup of  $\mathrm{SL}(2, \mathbb{R})$  which is commensurable with  $\Gamma$ , then  $G$  is conjugate to a subgroup of  $\Gamma_0(f)^+$  for some square-free  $f$ .*

As noted in the introduction, we may define the notion of a congruence subgroup for subgroups of  $\Gamma_0(f)^+$  using the same definition as for subgroups of  $\Gamma$ . However, it turns out to be equivalent, and more convenient, to introduce, following Thompson, the appropriate generalization of  $\Gamma(N)$ . Recall that

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

DEFINITION 2.3.  $G(n, f) = \Gamma_0(nf) \cap \Gamma(n)$ .

Note that  $G(n, f)$  is a normal subgroup of  $\Gamma_0(f)^+$  and that  $G(n, 1) = \Gamma(n)$ .

DEFINITION 2.4. *Call a subgroup  $G$  of  $\Gamma_0(f)^+$  a congruence subgroup if  $G(n, f) \subseteq G$  for some  $n$ .*

The following proposition shows that Definition 2.4 is equivalent, for subgroups of  $\Gamma_0(f)^+$ , to the standard definition of a congruence subgroup given in the introduction.

PROPOSITION 2.5. *A subgroup  $G$  of  $\mathrm{SL}(2, \mathbb{R})$  contains a subgroup  $G(n, f)$  for some  $n$  and some  $f$  if and only if  $G$  contains  $\Gamma(m)$  for some  $m$ .*

*Proof.* Suppose  $G$  contains  $\Gamma(m)$ . Then  $\Gamma(m) = G(m, 1)$  and so  $G$  contains  $G(m, 1)$ . Conversely, if  $G$  contains  $G(n, f)$ , then we have  $\Gamma(nf) \subseteq \Gamma_0(nf) \cap \Gamma(n)$  and so  $\Gamma(nf)$  is contained in  $G$ . □

We may also extend the notion of the level of a congruence subgroup.

DEFINITION 2.6. *If  $G$  is a congruence subgroup of  $\Gamma_0(f)^+$ , then let  $n = n(G, f)$  be the smallest positive integer such that  $G(n, f) \subseteq G$ . We call  $n(G, f)$  the level of  $G$ .*

If  $f = 1$ , then  $\Gamma_0(f)^+ = \Gamma$  and this definition of level coincides with the usual definition, since  $G(n, 1) = \Gamma(n)$ . However, if  $G$  is a subgroup of  $\Gamma_0(f)^+$  and  $f \neq 1$ , then it is not necessarily the case that  $n = n(G, f)$  is the smallest  $n$  such that  $\Gamma(n)$  is contained in  $G$ . Moreover, if  $G$  is a congruence subgroup of both  $\Gamma_0(f_1)^+$  and  $\Gamma_0(f_2)^+$ , with  $f_1 \neq f_2$ , then  $n(G, f_1)$  and  $n(G, f_2)$  are not necessarily equal.

### 3. The Algorithm

By Helling's Theorem, every subgroup of  $\mathrm{SL}(2, \mathbb{R})$  which is commensurable with  $\mathrm{SL}(2, \mathbb{Z})$  is conjugate to a subgroup of  $\Gamma_0(f)^+$  for some square-free, positive integer  $f$ . Thus to tabulate all conjugacy classes of congruence subgroups of genus-zero and genus-one, it is sufficient to list the genus-zero and genus-one subgroups of  $\Gamma_0(f)^+$  for the (finite) set of values of  $f$  such that  $\Gamma_0(f)^+$  is genus-zero or genus-one. This was done in [5]. However, in [5] the groups were found up to conjugacy in each  $\Gamma_0(f)^+$ . If a class occurred for two different values of  $f$ , then this was recorded in the tables, but the full  $\mathrm{SL}(2, \mathbb{R})$  conjugacy classes were not computed.

In this section we give an algorithm to find these classes. Table 1 records the results.

Suppose that  $K_1$  and  $K_2$  are subgroups of  $\Gamma_0(f_1)^+$  and  $\Gamma_0(f_2)^+$  respectively, and that  $K_1$  and  $K_2$  represent two of the classes in Table 2 of [5]. We want to test  $K_1$  and  $K_2$  for conjugacy in  $\mathrm{SL}(2, \mathbb{R})$ . It would initially seem that we have to consider the case  $f_1 \neq f_2$ . It turns out, however, that we do not need to consider such cases, since we can find a square-free integer  $f_{min}$  such that  $K_1$  and  $K_2$  are both subgroups of  $\Gamma_0(f_{min})^+$ .

PROPOSITION 3.1. *Suppose  $K_1$  and  $K_2$  are congruence subgroups of  $\Gamma_0(f_1)^+$  and  $\Gamma_0(f_2)^+$  respectively. If  $K_1$  and  $K_2$  are conjugate in  $\mathrm{SL}(2, \mathbb{R})$ , then there is some positive, square-free integer  $f_{min}$ , such that  $K_1$  and  $K_2$  are subgroups of  $\Gamma_0(f_{min})^+$ .*

*Proof.* First note that if  $m^{-1}K_1m = K_1^m = K_2$  for some  $m$  in  $\mathrm{SL}(2, \mathbb{R})$ , then  $m$  is a multiple of a primitive integer matrix, since  $K_1$  and  $K_2$  are commensurable with  $\mathrm{SL}(2, \mathbb{Z})$ . (This follows easily from the fact that  $K_1$  and  $K_2$  both have  $\mathbb{Q}^* = \mathbb{Q} \cup \{\infty\}$  as the set fixed by parabolic elements and so  $m$  must map  $\mathbb{Q}^*$  to  $\mathbb{Q}^*$ .) Next, for any element  $m$  of  $\mathrm{SL}(2, \mathbb{R})$  which is equal to  $\lambda A$ , where  $A$  is a primitive integer matrix, we define the normalized determinant  $\langle m \rangle$  to be the determinant of  $A$ . It is not difficult to verify that  $\langle m \rangle$  is well defined.

If  $B \in \mathrm{SL}(2, \mathbb{Z})$ , then  $\langle mB \rangle = \langle m \rangle$ . So the set of normalized determinants of the elements of  $K_1$  is finite, since  $K_1$  contains some  $\Gamma(N)$  with finite index. Let

$\{e_1, \dots, e_k\}$  be the set of normalized determinants of the elements of  $K_1$ . These must be square-free, since  $K_1$  is contained in  $\Gamma_0(f_1)^+$ , so the normalized determinants divide  $f_1$ . Then  $f_{\min} = \mathrm{lcm}(e_1, \dots, e_k)$  is the smallest square-free integer such that  $K_1$  is contained in  $\Gamma_0(f_{\min})^+$ . The matrices of  $K_2$  have the same normalized determinants as  $K_1$  up to squares, since they are obtained by conjugating by a multiple of a primitive integer matrix. But  $K_2$  is contained in some Helling group and so the normalized determinants of its elements must also be square-free. So the set of its normalized determinants is also  $\{e_1, \dots, e_k\}$ . Thus  $K_2$  is also contained in  $\Gamma_0(f_{\min})^+$ .  $\square$

So to find all  $\mathrm{SL}(2, \mathbb{R})$  conjugates of the congruence subgroups of interest, we can first find all  $\mathrm{SL}(2, \mathbb{R})$  conjugates within  $\Gamma_0(f)^+$  for each  $f$ . Then the full  $\mathrm{SL}(2, \mathbb{R})$  classes are obtained using the known intersections of classes for different values of  $f$  already computed in Table 2 of [5].

The area of a fundamental domain is invariant under conjugation in  $\mathrm{SL}(2, \mathbb{R})$  and so if  $K_1$  and  $K_2$  are subgroups of  $\Gamma_0(f)^+$  which are conjugate in  $\mathrm{SL}(2, \mathbb{R})$ , they have the same index in  $\Gamma_0(f)^+$ . Also, the number of classes of elliptic fixed points of given order and the cusp number are invariant under  $\mathrm{SL}(2, \mathbb{R})$  conjugation. Thus when testing  $K_1$  and  $K_2$  for  $\mathrm{SL}(2, \mathbb{R})$  conjugacy these invariants are first tested for equality. Also, by [5] Corollary 4.8, the set of primes dividing the levels of  $K_1$  and  $K_2$  are equal if the groups are conjugate in  $\mathrm{SL}(2, \mathbb{R})$ .

Given two groups which pass these initial tests, we now want to bound the number of conjugating matrices  $m$  which have to be considered. As  $\Gamma_0(f)^+$  acts transitively on  $\mathbb{Q} \cup \{\infty\}$ , by replacing  $K_2$  by a conjugate group in  $\Gamma_0(f)^+$  we can arrange for  $m$  to have the form  $m = \lambda \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}$ , with  $p, q, r$  integers such that  $\mathrm{gcd}(p, q, r) = 1$ ,  $pr > 0$  and  $\lambda = (pr)^{-1/2}$ . By [5] Proposition 4.7 we have the following constraints:  $p \mid \ell_1$ ,  $r \mid (\ell_1/p) \mathrm{gcd}(f, p)$ ,  $0 \leq q < p$  and also  $\ell_2 \mid \mathrm{gcd}(pr, \ell_1) \ell_1$ , where  $\ell_i$  is the level of  $K_i$ ,  $i = 1, 2$ . This assumes that we fix  $K_1$  and conjugate  $K_2$  in  $\Gamma_0(f)^+$ . If we also allow conjugations of  $K_1$  in  $\Gamma_0(f)^+$ , then we can also impose  $0 \leq q < \mathrm{gcd}(p, r)$ , since we can then conjugate  $K_1$  by  $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ . As  $K_2^{m^{-1}} = K_1$  we can apply the same arguments to get the conditions:  $r \mid \ell_2$  and  $p \mid (\ell_2/r) \mathrm{gcd}(f, r)$ , but again this assumes we allow a conjugation of  $K_1$  in  $\Gamma_0(f)^+$ . This produces a finite list of possible conjugating matrices  $m$ .

To summarize: given  $K_1$  and  $K_2$ , subgroups of  $\Gamma_0(f)^+$ , we first test the obvious invariants for equality. Then, given the levels of the two groups, we can find a finite list of possible conjugating matrices with the property that, if  $K_1$  and  $K_2$  are conjugate in  $\mathrm{SL}(2, \mathbb{R})$ , then there is some  $m$  in the list and some conjugate  $K'_1$  of  $K_1$  in  $\Gamma_0(f)^+$  and some conjugate  $K'_2$  of  $K_2$  in  $\Gamma_0(f)^+$  such that  $m$  conjugates  $K'_1$  to  $K'_2$ . Since there are only finitely many conjugates  $K'_1$  and  $K'_2$ , this leads to a finite number of cases to test and so we have an algorithm for finding all the  $\mathrm{SL}(2, \mathbb{R})$  conjugacy classes of congruence subgroups of a given genus.

Since the groups are infinite there is still the problem of giving an algorithm to perform the test for equality. This can be done as follows. It is not difficult to verify that if  $m = \lambda \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}$  as above, then  $mG(pr\ell_1, f)m^{-1} \subseteq G(\ell_1, f) \subseteq K_1$ . Let  $\ell_3 = \mathrm{lcm}(pr\ell_1, \ell_2)$ , then since  $G(\ell_3, f) \subseteq G(pr\ell_1)$ , we have  $mG(\ell_3, f)m^{-1} \subseteq K_1$ .

Moreover,  $G(\ell_3, f) \subseteq G(\ell_2, f) \subseteq K_2$ . Thus to test if  $mK_2m^{-1}$  is equal to  $K_1$ , it is sufficient to test if  $msm^{-1}$  is in  $K_1$ , for all  $s \in S$  for some set of generators  $S$  of  $K_2$  over  $G(\ell_3, f)$ . (The fact that  $G(\ell_3, f)$  and  $G(\ell_2, f)$  are normal subgroups of  $K_2$  and that  $K_2/G(\ell_2, f)$  was constructed as a permutation group in [5] simplifies the computation of the cosets). Thus we are finished if we have an algorithm for testing when an element  $g$  in  $\Gamma_0(f)^+$  is in  $K_1$ , but as the group  $K_1/G(\ell_1, f)$  was constructed as a subgroup of  $\Gamma_0(f)^+/G(\ell_1, f)$  in [5], we can simply test to see if the image of  $g$  in  $\Gamma_0(f)^+/G(\ell_1, f)$  is in  $K_1/G(\ell_1, f)$ .

In summary we have the following algorithm.

### The Algorithm

**Input:** A square-free integer  $f$ , and two congruence subgroups  $K_1$  and  $K_2$  of  $\Gamma_0(f)^+$ , of levels  $\ell_1$  and  $\ell_2$  respectively.

**Output:** Return **true** if  $K_1$  and  $K_2$  are conjugate in  $\mathrm{SL}(2, \mathbb{R})$  and **false** otherwise.

- 
- $I_1 \leftarrow$  invariants of  $K_1$
  - $I_2 \leftarrow$  invariants of  $K_2$
  - If  $I_1 \neq I_2$  then return **false**
  - Else
    - $C_1 \leftarrow$  conjugates of  $K_1$  in  $\Gamma_0(f)^+$
    - $C_2 \leftarrow$  conjugates of  $K_2$  in  $\Gamma_0(f)^+$
    - $M \leftarrow$  list of possible conjugating matrices
    - For  $K$  in  $C_2$ 
      - For  $m = \begin{pmatrix} p & q \\ 0 & r \end{pmatrix}$  in  $M$ 
        - $\ell_3 \leftarrow \mathrm{lcm}(pr\ell_1, \ell_2)$ .
        - $S \leftarrow$  a set of generators of  $K$  over the normal subgroup  $G(\ell_3, f)$ .
        - $S' \leftarrow mSm^{-1}$
        - If  $S'$  is not a subset of  $\Gamma_0(f)^+$  then move to the next  $m$
        - For each  $L$  in  $C_1$ 
          - If the image of  $S'$  in  $\Gamma_0(f)^+/G(\ell_1, f)$  is contained in  $L/G(\ell_1, f)$  then return **true**
  - If after testing all elements of  $C_2$  we have not returned **true** then return **false**
- 

### 4. Conclusions

In the previous section an algorithm was given for determining whether two subgroups of  $\Gamma_0(f)^+$  are conjugate in  $\mathrm{SL}(2, \mathbb{R})$ . Table 2 of [5] records when the conjugacy classes of subgroups  $C_i$  of  $\Gamma_0(f_i)^+$ ,  $i = 1, 2$ , are such that  $f_1 \neq f_2$  and  $C_1 \cap C_2$  is not empty. As explained in the last section, combining these two pieces

of information yields the  $\mathrm{SL}(2, \mathbb{R})$  conjugacy classes of congruence subgroups of genus-zero and genus-one.

The algorithm of the previous section was programmed in MAGMA [1]. The results are presented in Table 1. Each entry in this table is a list of the  $\Gamma_0(f)^+$  conjugacy classes from Table 2 of [5] which are subsets of one  $\mathrm{SL}(2, \mathbb{R})$  conjugacy class. Cases with a single  $\Gamma_0(f)^+$  conjugacy class have been omitted. The notation is as in Table 2 of [5]: each class has a label  $level(letter)_f^{genus}$ , where  $f$  is a square-free integer such that the group is contained in  $\Gamma_0(f)^+$ ;  $level$  is the level of the group with respect to this  $f$  (as defined in Definition 2.4);  $genus$  is its genus; and  $letter$  is a letter labelling the group amongst all groups of the same level and genus.

Sebbar [8] has shown that there are 15  $\mathrm{SL}(2, \mathbb{R})$  conjugacy classes of torsion-free congruence subgroups of genus-zero. As a test of the results of this paper, using the additional data from Table 2 of [5], we also find 15  $\mathrm{SL}(2, \mathbb{R})$  conjugacy classes of torsion-free, genus-zero congruence subgroups. Moreover, these give rise to 33 classes of subgroups of the modular group, again in agreement with Sebbar's results.

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*On Conjugacy Classes of Congruence Subgroups of  $\mathrm{PSL}(2, \mathbb{R})$ .*

Table 1:  $\mathrm{PSL}(2, \mathbb{R})$  conjugacy classes of congruence subgroups of genus-zero and genus-one.

Classes	Classes	Classes	Classes
$2B_1^0 \ 1B_2^0$	$8C_1^0 \ 4D_2^0$	$14C_1^0 \ 2D_7^0$	$4H_3^0 \ 4I_3^0 \ 8C_3^0$ $2L_6^0 \ 4I_6^0$
$2C_1^0 \ 4B_1^0 \ 2D_2^0$	$8J_1^0 \ 4K_2^0$	$14D_1^0 \ 14C_7^0$	$4J_3^0 \ 8D_3^0 \ 4J_6^0$
$3B_1^0 \ 1B_3^0$	$8J_1^0 \ 8K_1^0 \ 16E_1^0$ $8L_2^0 \ 8U_2^0 \ 8V_2^0$	$15C_1^0 \ 3C_5^0$	$4K_3^0 \ 8F_3^0 \ 4M_6^0$
$3D_1^0 \ 9B_1^0 \ 3D_3^0$	$8L_1^0 \ 16F_1^0 \ 8W_2^0$	$15D_1^0 \ 15E_5^0$	$6E_3^0 \ 3C_6^0$
$4C_1^0 \ 2C_2^0$	$8M_1^0 \ 8N_2^0$	$16C_1^0 \ 8J_2^0$	$7A_3^0 \ 1C_{21}^0$
$4E_1^0 \ 8D_1^0 \ 2E_2^0$ $4F_2^0$	$8P_1^0 \ 8Q_1^0 \ 16H_1^0$ $16I_1^0 \ 32A_1^0 \ 8AB_2^0$ $8Y_2^0 \ 8Z_2^0 \ 16E_2^0$	$18C_1^0 \ 6F_3^0$ $24A_1^0 \ 12E_2^0$	$7B_3^0 \ 7B_{21}^0$
$4F_1^0 \ 8E_1^0 \ 4G_2^0$		$26A_1^0 \ 26C_{13}^0$	$12F_3^0 \ 24B_3^0 \ 12D_6^0$
$4G_1^0 \ 8H_1^0 \ 16D_1^0$ $4J_2^0 \ 8M_2^0$	$8R_1^0 \ 8AA_2^0$	$28A_1^0 \ 28C_7^0$	$12G_3^0 \ 12H_3^0 \ 12I_3^0$ $24A_3^0 \ 12A_6^0 \ 12B_6^0$ $12C_6^0$
$5B_1^0 \ 1B_5^0$	$9C_1^0 \ 3C_3^0$	$30A_1^0 \ 30C_5^0$	$12N_3^0 \ 6F_6^0$
$5D_1^0 \ 5C_5^0$	$9H_1^0 \ 9I_1^0 \ 27A_1^0$ $9F_3^0 \ 9G_3^0$	$36A_1^0 \ 12C_3^0$	$12R_3^0 \ 12S_3^0 \ 24D_3^0$ $24E_3^0 \ 12M_6^0 \ 12N_6^0$ $12O_6^0$
$5G_1^0 \ 25A_1^0 \ 5H_5^0$	$9J_1^0 \ 9E_3^0$	$48A_1^0 \ 24C_2^0$	$21A_3^0 \ 21B_3^0 \ 21B_{21}^0$
$5H_1^0 \ 25B_1^0 \ 5I_5^0$	$10B_1^0 \ 10D_5^0$	$3B_2^0 \ 1C_6^0$	$2B_5^0 \ 1B_{10}^0$
$6C_1^0 \ 2D_3^0$	$10C_1^0 \ 5D_2^0 \ 2C_5^0$ $1E_{10}^0$	$3H_2^0 \ 9B_2^0 \ 3E_6^0$	$2D_5^0 \ 4B_5^0 \ 2C_{10}^0$
$6D_1^0 \ 3C_2^0$	$10F_1^0 \ 5G_2^0 \ 10N_5^0$ $5C_{10}^0$	$5B_2^0 \ 1D_{10}^0$ $6D_2^0 \ 2E_6^0$	$3B_5^0 \ 1B_{15}^0$
$6F_1^0 \ 3F_2^0 \ 2F_3^0$ $1E_6^0$	$10G_1^0 \ 20A_1^0 \ 10D_2^0$ $10Q_5^0 \ 20E_5^0 \ 10G_{10}^0$	$6M_2^0 \ 2M_6^0$ $9F_2^0 \ 9G_2^0 \ 9C_6^0$	$4C_5^0 \ 2D_{10}^0$
$6G_1^0 \ 3G_2^0$		$10B_2^0 \ 2F_{10}^0$	$4E_5^0 \ 2I_{10}^0$
$6H_1^0 \ 12D_1^0 \ 6H_2^0$	$12B_1^0 \ 12C_3^0$	$10E_2^0 \ 10J_{10}^0$	$6B_5^0 \ 2D_{15}^0$
$6I_1^0 \ 12E_1^0 \ 6L_2^0$ $2H_3^0 \ 4G_3^0 \ 2K_6^0$	$12C_1^0 \ 6E_2^0$ $12H_1^0 \ 6N_2^0$	$12D_2^0 \ 4E_6^0$	$8A_5^0 \ 4D_{10}^0$
$6J_1^0 \ 18D_1^0 \ 6C_3^0$	$12I_1^0 \ 12J_1^0 \ 24B_1^0$ $12G_2^0 \ 12H_2^0 \ 12F_3^0$ $12Q_3^0 \ 24C_3^0 \ 12I_6^0$ $12J_6^0$	$12I_2^0 \ 12L_6^0$ $2C_3^0 \ 1D_6^0$	$10F_5^0 \ 10G_5^0 \ 5A_{10}^0$
$6K_1^0 \ 18F_1^0 \ 3I_2^0$ $9E_2^0 \ 6H_3^0 \ 3C_6^0$		$10I_5^0 \ 20A_5^0 \ 10B_{10}^0$	$10H_5^0 \ 20A_5^0 \ 10B_{10}^0$
$6L_1^0 \ 12G_1^0 \ 6O_2^0$	$13A_1^0 \ 1B_{13}^0$	$2E_3^0 \ 4B_3^0 \ 2H_6^0$	$10I_5^0 \ 20B_5^0 \ 10A_{10}^0$
$7C_1^0 \ 1B_7^0$	$13B_1^0 \ 13D_{13}^0$	$2G_3^0 \ 4C_3^0 \ 2C_6^0$	$10C_5^0 \ 20G_5^0 \ 10F_{10}^0$
$7G_1^0 \ 7B_7^0$	$13C_1^0 \ 13E_{13}^0$	$4D_3^0 \ 2F_6^0$	$10P_5^0 \ 20F_5^0 \ 10E_{10}^0$

On Conjugacy Classes of Congruence Subgroups of  $\text{PSL}(2, \mathbb{R})$ .

Classes			Classes			Classes			Classes		
$15C_5^0$	$15D_5^0$	$5A_{15}^0$	$8G_1^1$	$16G_1^1$	$8AC_2^1$ $8V_2^1$	$12L_1^1$	$6L_2^1$		$16D_1^1$	$8N_2^1$	
$15G_5^0$	$15H_5^0$	$15A_{15}^0$				$12M_1^1$	$6K_2^1$		$16F_1^1$	$8W_2^1$	
$20H_5^0$	$20I_5^0$	$10I_{10}^0$	$8H_1^1$	$16I_1^1$	$8AE_2^1$						
	$2C_7^0$	$1B_{14}^0$	$8I_1^1$	$16H_1^1$	$8AD_2^1$						
$2E_7^0$	$4A_7^0$	$2D_{14}^0$	$8K_1^1$	$16M_1^1$	$32E_1^1$ $8AH_2^1$	$12P_1^1$	$36B_1^1$	$12K_3^1$	$16L_1^1$	$32D_1^1$	$16R_2^1$
	$6B_7^0$	$3B_{14}^0$		$9A_1^1$	$3A_3^1$	$12Q_1^1$ $12V_2^1$	$24I_1^1$ $4H_3^1$	$6P_2^1$ $8J_3^1$	$17A_1^1$	$1A_{17}^1$	
	$14B_7^0$	$7C_{14}^0$	$9D_1^1$	$27A_1^1$	$3B_3^1$ $9D_3^1$		$2H_6^1$	$4U_6^1$	$17B_1^1$	$17A_{17}^1$	
	$2B_{11}^0$	$1B_{22}^0$		$9E_1^1$	$9B_3^1$		$12S_1^1$	$12O_3^1$	$17C_1^1$	$17B_{17}^1$	
	$3A_{11}^0$	$1B_{33}^0$					$12U_1^1$	$6R_2^1$	$18D_1^1$	$6E_3^1$	
	$4B_{11}^0$	$2C_{22}^0$	$9J_1^1$	$27C_1^1$	$9H_3^1$ $9I_3^1$		$12V_1^1$	$24K_1^1$	$12Y_2^1$	$18E_1^1$	$9D_2^1$
	$2C_{15}^0$	$1E_{30}^0$		$10A_1^1$	$2B_5^1$	$12W_1^1$ $12AD_2^1$	$24L_1^1$ $12T_3^1$	$12AC_2^1$ $24O_3^1$	$18F_1^1$	$9I_2^1$	
	$2E_{15}^0$	$1F_{30}^0$		$10B_1^1$	$5A_2^1$		$12U_6^1$	$12V_6^1$	$18K_1^1$	$6L_3^1$	
$2F_{15}^0$	$4A_{15}^0$	$2D_{30}^0$		$10D_1^1$	$10C_5^1$		$14C_1^1$	$7E_2^1$	$19A_1^1$	$1A_{19}^1$	
	$2A_{23}^0$	$1B_{46}^0$		$10F_1^1$	$5C_2^1$		$14D_1^1$	$7B_2^1$	$19B_1^1$	$19A_{19}^1$	
$6C_1^1$	$12B_1^1$	$6C_2^1$	$10G_1^1$ $2D_5^1$	$20E_1^1$ $4D_5^1$	$10F_2^1$ $2H_{10}^1$	$14E_1^1$	$7F_2^1$	$2A_7^1$ $1B_{14}^1$	$20B_1^1$	$20C_1^1$	$20E_5^1$
$6D_1^1$	$18C_1^1$	$6D_3^1$		$10I_1^1$	$5D_2^1$	$14L_1^1$	$7N_2^1$	$14C_7^1$ $7B_{14}^1$	$20D_1^1$	$4C_5^1$	
$6E_1^1$	$12K_1^1$	$6M_2^1$	$10K_1^1$ $10F_5^1$	$20J_1^1$ $20W_5^1$	$10L_2^1$ $10L_{10}^1$		$15B_1^1$	$5P_3^1$	$20F_1^1$	$10E_2^1$	$4E_5^1$ $2G_{10}^1$
$6F_1^1$ $36C_1^1$ $6K_3^1$	$12T_1^1$ $6Q_2^1$ $12P_3^1$	$18J_1^1$ $18F_2^1$ $6I_6^1$		$11A_1^1$	$1A_{11}^1$	$15C_1^1$	$5C_3^1$	$3B_5^1$ $1B_{15}^1$	$20I_1^1$	$20V_5^1$	
$7B_1^1$	$49A_1^1$	$7A_7^1$		$11D_1^1$	$11A_{11}^1$		$15E_1^1$	$3C_5^1$	$20K_1^1$ $40B_1^1$ $20AA_5^1$	$20L_1^1$ $20G_2^1$ $40G_5^1$	$40A_1^1$ $20H_2^1$ $20L_{10}^1$
	$8A_1^1$	$4B_2^1$		$12A_1^1$	$4B_3^1$	$15G_1^1$	$5F_3^1$	$15E_5^1$ $5A_{15}^1$	$20M_1^1$	$10K_2^1$	$20Y_5^1$ $10K_{10}^1$
$8B_1^1$	$16A_1^1$	$4D_2^1$ $8H_2^1$	$12F_1^1$	$6I_2^1$	$4C_3^1$ $2G_6^1$		$15H_1^1$	$15F_5^1$	$21A_1^1$	$21B_1^1$	$21A_7^1$
	$8C_1^1$	$4E_2^1$		$12I_1^1$	$4G_3^1$		$15I_1^1$	$15D_3^1$ $15A_{15}^1$	$21C_1^1$	$7A_3^1$	$3C_7^1$ $1B_{21}^1$
$8D_1^1$	$16C_1^1$	$8T_2^1$									
$8F_1^1$ $4F_2^1$	$16E_1^1$ $8X_2^1$	$32A_1^1$ $16I_2^1$	$12J_1^1$	$24E_1^1$	$12H_2^1$	$16B_1^1$	$8G_2^1$		$21I_1^1$	$21C_3^1$	$21C_7^1$ $21B_{21}^1$

*On Conjugacy Classes of Congruence Subgroups of  $\text{PSL}(2, \mathbb{R})$ .*

Classes	Classes	Classes	Classes
$24F_1^1$ $12F_2^1$	$13A_2^1$ $1B_{26}^1$	$8L_3^1$ $4R_6^1$	$3D_5^1$ $9B_5^1$ $3B_{15}^1$
$24J_1^1$ $12X_2^1$	$13B_2^1$ $13B_{26}^1$	$8M_3^1$ $4T_6^1$	$4B_5^1$ $2C_{10}^1$
$26A_1^1$ $2C_{13}^1$	$15D_2^1$ $3F_{10}^1$	$8N_3^1$ $16F_3^1$ $8P_6^1$	$4F_5^1$ $8B_5^1$ $4K_{10}^1$
$26B_1^1$ $26F_{13}^1$	$15E_2^1$ $5C_6^1$	$8O_3^1$ $16G_3^1$ $8Q_6^1$	$4G_5^1$ $8C_5^1$ $4J_{10}^1$
$27B_1^1$ $9C_3^1$	$17A_2^1$ $1C_{34}^1$	$8P_3^1$ $16H_3^1$ $8V_6^1$	$6D_5^1$ $3D_{10}^1$
$30D_1^1$ $6E_5^1$	$17B_2^1$ $17C_{34}^1$	$10B_3^1$ $5B_6^1$	$6H_5^1$ $3G_{10}^1$ $2F_{15}^1$ $1F_{30}^1$
$30E_1^1$ $30K_5^1$	$20B_2^1$ $20C_2^1$ $20C_{10}^1$	$12A_3^1$ $24B_3^1$ $12A_6^1$	$7A_5^1$ $1A_{35}^1$
$32B_1^1$ $16G_2^1$	$20D_2^1$ $4I_{10}^1$	$12D_3^1$ $24A_3^1$ $12B_6^1$	$7B_5^1$ $7A_{35}^1$
$39A_1^1$ $39B_1^1$ $39A_{13}^1$	$20I_2^1$ $20J_2^1$ $20P_{10}^1$	$12H_3^1$ $24G_3^1$ $12H_6^1$ $12K_6^1$	$8A_5^1$ $4F_{10}^1$
$52A_1^1$ $52B_1^1$ $52B_{13}^1$	$24J_2^1$ $24L_2^1$ $8M_6^1$	$12I_3^1$ $24H_3^1$ $12I_6^1$ $12J_6^1$	$8E_5^1$ $4N_{10}^1$
$5E_2^1$ $25A_2^1$ $5A_{10}^1$	$24K_2^1$ $24M_2^1$ $24E_6^1$	$12L_3^1$ $24I_3^1$ $24J_3^1$ $24K_3^1$ $12M_6^1$ $12N_6^1$ $12O_6^1$	$10D_5^1$ $20H_5^1$ $10E_{10}^1$ $10E_5^1$ $20I_5^1$ $10F_{10}^1$
$6B_2^1$ $2B_6^1$	$24N_2^1$ $24O_2^1$ $24H_6^1$	$12M_3^1$ $24L_3^1$ $48A_3^1$ $12L_6^1$ $24G_6^1$	$16A_5^1$ $8F_{10}^1$ $20A_5^1$ $20B_5^1$ $10A_{10}^1$
$7A_2^1$ $1A_{14}^1$	$4A_3^1$ $2D_6^1$	$12Q_3^1$ $24N_3^1$ $12T_6^1$	$20AB_5^1$ $40H_5^1$ $20M_{10}^1$
$7G_2^1$ $7A_{14}^1$	$4D_3^1$ $2E_6^1$	$12S_3^1$ $24P_3^1$ $12W_6^1$	$20AC_5^1$ $40I_5^1$ $20K_{10}^1$
$9A_2^1$ $3C_6^1$	$4E_3^1$ $8C_3^1$ $2F_6^1$ $4M_6^1$	$13A_3^1$ $1A_{39}^1$	$20J_5^1$ $20Q_5^1$ $10I_{10}^1$
$9B_2^1$ $3D_6^1$	$4F_3^1$ $8D_3^1$ $4N_6^1$	$13B_3^1$ $13F_{39}^1$	$20K_5^1$ $10C_{10}^1$
$9G_2^1$ $9D_6^1$	$4I_3^1$ $8K_3^1$ $16E_3^1$ $4V_6^1$ $8O_6^1$	$14A_3^1$ $2D_{21}^1$	$20L_5^1$ $20O_5^1$ $10H_{10}^1$
$9H_2^1$ $9E_6^1$		$14B_3^1$ $2E_{21}^1$	$20M_5^1$ $20N_5^1$ $10J_{10}^1$
$9J_2^1$ $27A_2^1$ $9H_6^1$	$5A_3^1$ $1A_{15}^1$	$18J_3^1$ $9G_6^1$	$20P_5^1$ $10D_{10}^1$
$10B_2^1$ $2B_{10}^1$	$6C_3^1$ $3B_6^1$	$21A_3^1$ $21B_3^1$ $21A_{21}^1$	$20R_5^1$ $40C_5^1$ $20D_{10}^1$
$11A_2^1$ $1A_{22}^1$	$6F_3^1$ $12F_3^1$ $6D_6^1$	$21D_3^1$ $21E_3^1$ $21D_{21}^1$	$20S_5^1$ $40D_5^1$ $20E_{10}^1$
$11B_2^1$ $11A_{22}^1$	$6H_3^1$ $12G_3^1$ $6C_6^1$	$42A_3^1$ $42B_3^1$ $42B_{21}^1$	$20X_5^1$ $20Z_5^1$ $10M_{10}^1$
$12E_2^1$ $4K_6^1$	$6J_3^1$ $3E_6^1$	$2C_5^1$ $4A_5^1$ $2E_{10}^1$	$30C_5^1$ $30D_5^1$ $10A_{15}^1$
$12U_2^1$ $4S_6^1$	$8E_3^1$ $4J_6^1$		
$12W_2^1$ $4X_6^1$	$8F_3^1$ $4E_6^1$		

On Conjugacy Classes of Congruence Subgroups of  $\text{PSL}(2, \mathbb{R})$ .

Classes			Classes		
$30I_5^1$	$30J_5^1$	$10C_{15}^1$	$33A_{11}^1$	$33B_{11}^1$	$33A_{33}^1$
$40A_5^1$	$40B_5^1$	$20B_{10}^1$		$2B_{13}^1$	$1A_{26}^1$
$40E_5^1$	$40F_5^1$	$20J_{10}^1$		$3A_{14}^1$	$1B_{42}^1$
	$5A_6^1$	$1C_{30}^1$		$3D_{14}^1$	$1F_{42}^1$
	$5D_6^1$	$1D_{30}^1$	$2E_{15}^1$	$4C_{15}^1$	$2M_{30}^1$
	$5I_6^1$	$5C_{30}^1$	$2G_{15}^1$	$4B_{15}^1$	$2N_{30}^1$
	$7A_6^1$	$1C_{42}^1$	$2H_{15}^1$	$4G_{15}^1$	$2R_{30}^1$
	$7B_6^1$	$7A_{42}^1$		$4D_{15}^1$	$2I_{30}^1$
$2B_7^1$	$4B_7^1$	$2E_{14}^1$		$4E_{15}^1$	$2H_{30}^1$
	$3A_7^1$	$1A_{21}^1$		$4H_{15}^1$	$2Q_{30}^1$
	$4C_7^1$	$2C_{14}^1$		$6D_{15}^1$	$3C_{30}^1$
	$4D_7^1$	$2D_{14}^1$		$10D_{15}^1$	$5D_{30}^1$
$4F_7^1$	$8A_7^1$	$2G_{14}^1$ $4E_{14}^1$		$2A_{17}^1$	$1B_{34}^1$
				$3B_{17}^1$	$1A_{51}^1$
$6E_7^1$	$12A_7^1$	$6E_{14}^1$		$2B_{19}^1$	$1A_{38}^1$
	$6F_7^1$	$3E_{14}^1$		$2C_{21}^1$	$1A_{42}^1$
$14D_7^1$	$28F_7^1$	$14C_{14}^1$		$3B_{22}^1$	$1A_{66}^1$
$28A_7^1$	$28B_7^1$	$28C_7^1$			
$56A_7^1$	$28A_{14}^1$	$28B_{14}^1$ $28C_{14}^1$	$2C_{23}^1$	$4A_{23}^1$	$2C_{46}^1$
	$3A_{10}^1$	$1B_{30}^1$		$3A_{23}^1$	$1A_{69}^1$
				$2C_{31}^1$	$1A_{62}^1$
$2B_{11}^1$	$4B_{11}^1$	$2B_{22}^1$		$2C_{33}^1$	$1C_{66}^1$
$2C_{11}^1$	$4A_{11}^1$	$2C_{22}^1$		$2C_{35}^1$	$1C_{70}^1$
$4D_{11}^1$	$8C_{11}^1$	$4C_{22}^1$		$3A_{35}^1$	$1C_{105}^1$
	$5A_{11}^1$	$1A_{55}^1$		$2E_{39}^1$	$1B_{78}^1$
	$5B_{11}^1$	$5A_{55}^1$		$2A_{47}^1$	$1A_{94}^1$
	$6A_{11}^1$	$2D_{33}^1$		$2C_{55}^1$	$1A_{110}^1$

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