

REPRESENTATION-DIRECTED DIAMONDS

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Abstract

A module over a finite-dimensional algebra is called a ‘diamond’ if it has a simple top and a simple socle. Using covering theory, the classification of all diamonds for algebras of finite representation type over algebraically closed fields can be reduced to representation-directed algebras. We prove a criterion referring to the positive roots of the corresponding Tits quadratic form, which makes it easy to check whether a representation-directed algebra has a faithful diamond. Using an implementation of this criterion in the CREP program system on representation theory, we are able to classify all exceptional representation-directed algebras having a faithful diamond. We obtain a list of 157 algebras up to isomorphism and duality. The 52 maximal members of this list are presented at the end of this paper.

1. *Introduction and main result*

Following the conventions used in [11], a (right) module D over an associative ring A is said to be a *diamond*, provided that it has an essential simple submodule and a superfluous maximal submodule. Obviously, any diamond is indecomposable. If A happens to be a finite-dimensional algebra over a field k , then a module D is a diamond if and only if D is a finite-dimensional module with a simple socle and a simple top. (Recall that the top of a module is the factor module by the Jacobson radical.) Since any indecomposable module of length 2 is a diamond, a finite-dimensional algebra A will usually have infinitely many isomorphism classes of diamonds. On the other hand, an algebra A of finite representation type (that is, A has only finitely many indecomposable modules up to isomorphism) can have only finitely many isomorphism classes of diamonds. At least, if the field k is algebraically closed, the algebras of finite representation type have been well studied. We refer to [9] for an introduction to this theory.

Using the covering theory developed in [3], we may reduce the study of modules over finite-dimensional algebras A of finite representation type over an algebraically closed field k to the case that A is representation-directed. In particular, any diamond over an algebra of finite representation type is obtained from a diamond over a representation-directed algebra by the application of the push-down functor associated with the universal Galois covering. Recall that, following [10], an algebra A is said to be *representation-directed* if there exists no sequence X_0, \dots, X_n of indecomposable finite-dimensional A -modules with $n > 0$ and $X_0 \cong X_n$ such that for each $i = 1, \dots, n$ there is a homomorphism $X_{i-1} \rightarrow X_i$ which is neither an isomorphism nor zero.

Since factor algebras of representation-directed algebras are representation-directed, in order to find all diamonds over representation-directed algebras it suffices to look at

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all representation-directed algebras which have faithful indecomposable modules, and to check which of these modules are diamonds. Fortunately, all the representation-directed algebras over an algebraically closed field which have an indecomposable faithful module have been classified. They appear in 24 families (see [1]), together with many exceptions in low dimensions (see [4]). These exceptions were found by a computer program, and are accessible via a data base in the CREP system (see [6]). Hence, it remains only to find out which of the algebras appearing in the families and in the data base have a faithful indecomposable module which is a diamond. It is the aim of this paper to present a convenient criterion to determine when this happens.

Theorem. *Let A be a representation-directed algebra over an algebraically closed field k . Then A is obtained from a representation-directed algebra having a faithful diamond by a reorientation of the arms if and only if the vector $\varepsilon = (1, \dots, 1)$ is the only sincere positive 1-root of the Tits form of A .*

We explain all the notation in the next section. Here, we note that the lists mentioned above give, for each algebra, the maximal (with respect to the natural product order on \mathbb{Z}^r) positive roots of the associated Tits form. It is observed in [11] that, of the 24 families, only those which are listed in [10] as (Bo1), (Bo15), (Bo16), (Bo17), (Bo19), (Bo20) and (Bo21) have a faithful diamond. An implementation of the above criterion, which searches the data base in the CREP system, yields a list of 157 exceptional algebras (up to isomorphism and duality) having a faithful diamond. As we shall explain in the next section, any such algebra is the incidence algebra of a finite, partially ordered set. In order to present our list in a compact way, we display the Hasse diagrams of its 52 maximal members in Figures 2 and 3, at the end of this paper.

2. Representation-directed algebras

We use the terminology of [10]. For the study of diamonds, we may assume, without loss of generality, that our given algebra A is basic and connected. It is well-known (see [8]) that any basic finite-dimensional algebra A over an algebraically closed field k up to isomorphism can be written as $k\bar{\Delta}/I$, where $\bar{\Delta}$ is a finite quiver and I is an admissible ideal of the path algebra $k\bar{\Delta}$. We denote by $a(x, y)$ the number of arrows from x to y in $\bar{\Delta}$, and by $b(x, y)$ the number of minimal generators of I starting in x and ending in y . After labelling the vertices of $\bar{\Delta}$ by $1, \dots, r$ we obtain a quadratic form $q_A : \mathbb{Z}^r \rightarrow \mathbb{Z}$, called the *Tits form*, given by $q_A(x) = \sum_{i=1}^r x_i^2 - \sum_{i,j=1}^r a(i, j)x_i x_j + \sum_{i,j=1}^r b(i, j)x_i x_j$ for $x = (x_1, \dots, x_r) \in \mathbb{Z}^r$. If A is representation-directed, then by [2] q_A is *weakly positive* (that is, $q_A(x) > 0$ for all $0 \neq x \in \mathbb{Z}^r$ with non-negative entries). Consequently, in this case q_A has only finitely many *positive 1-roots*; that is, vectors $x \in \mathbb{Z}^r$ with non-negative entries satisfying $q_A(x) = 1$.

The A -modules can be identified with the contravariant representations X of $\bar{\Delta}$, such that $X(\varrho) = 0$ for all elements ϱ of I (see [10]). Using this identification, the *dimension vector* $\mathbf{dim} X \in \mathbb{Z}^r$ is given by $(\mathbf{dim} X)_i = \dim_k X(i)$ for all vertices $i = 1, \dots, r$. By [2], for representation-directed A , the map \mathbf{dim} yields a bijection from the set of isomorphism classes of indecomposable A -modules to the set of positive 1-roots of q_A . A vector x in \mathbb{Z}^r is called *sincere* if $x_i \neq 0$ for all $i = 1, \dots, r$. Analogously, an A -module X is called *sincere* provided that $X(i) \neq 0$ for all $i = 1, \dots, r$. Thus the map \mathbf{dim} also yields a bijection between the set of isomorphism classes of sincere indecomposable A -modules and the set of sincere positive 1-roots of q_A .

It is well-known (see, for example, [10]) that an indecomposable module over a representation-directed algebra is faithful if and only if it is sincere. Moreover, it is shown in [10] that a representation-directed algebra which has an indecomposable sincere module is *simply connected* (see [3]). Hence A is *completely separating* in the terminology of [5], and can therefore be written as kS/J , where S is a finite partially ordered set and J is an ideal of the incidence algebra kS generated by elements (y, x) such that there is z in S satisfying $y < z < x$. Recall that the incidence algebra kS is the vector space with the basis given by all pairs (y, x) such that $y \leq x$ in S . The product $(z, y)(y', x)$ in kS is (z, x) for $y = y'$, and 0 otherwise. For $A = kS/J$, the quiver $\bar{\Delta}$ of A is the Hasse diagram of S , and we can also write A as $A = k\bar{\Delta}/I$, where I is the ideal of $k\bar{\Delta}$ generated by all differences $u - v$ of paths in $\bar{\Delta}$ with the same origin and terminus, together with all paths w starting in x and ending in y such that there is a generator (y, x) of J .

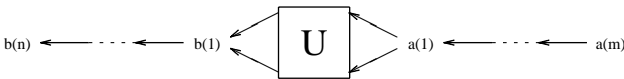
Let A be an algebra of the shape $A = kS/J$ for a finite partially ordered set S . With any subset T of S which is convex and *relation-free* (that is, for each generator (y, x) of J , x and y may not both lie in T) there is an associated *indicator module* δ_T , which is defined by $\delta_T(x) = k$ for all $x \in T$, and $\delta_T(x) = 0$ otherwise. Moreover, the arrows $\alpha : x \rightarrow y$ of $\bar{\Delta}$ are sent to the identity of k for x and y in T , and 0 otherwise.

If S is a finite partially ordered set with a unique maximal and a unique minimal element, then the algebra kS has a sincere diamond, namely δ_S . The following proposition establishes the converse of this observation for representation-directed algebras.

Proposition 1. *If $A = kS/J$ is a representation-directed algebra with a sincere diamond X , then $J = 0$, S has a unique minimal and a unique maximal element, X is isomorphic to δ_S , and X is up to isomorphism the only sincere indecomposable A -module.*

Proof. Since X is a diamond, there is an epimorphism $\phi : P(x) \rightarrow X$ for some element x of S where $P(x)$ comprises the indecomposable projective modules associated with x . It is easy to see that $P(x) \cong \delta_T$, where T is the subset of all y in S such that $y \geq x$ and $(x, y) \notin J$. The sincerity of X shows that $S = \text{supp } X \subseteq \text{supp } P(x) = T$. (For an A -module X , we denote by ‘ $\text{supp } X$ ’ the set of all elements y of S with $X(y) \neq 0$.) Hence x is the unique minimal element of S , and moreover $J = 0$ because $T = S$ is relation-free. Dually, S has a unique maximal element. If ϕ were not an isomorphism, then its kernel would be non-zero, and X would not be sincere. Finally, let N be another sincere indecomposable A -module. Since X is projective and dually also injective, there exist non-zero homomorphisms $X \rightarrow N$ and $N \rightarrow X$. Consequently, X and N have to be isomorphic because A is representation-directed. □

The above proposition shows that, if $A = kS$ is a representation-directed algebra with a sincere diamond, then there are two possible cases. Either S is a finite chain, or the Hasse diagram $\bar{\Delta}$ of S has the shape shown below, where $a(1)$ has at least two lower neighbors, $b(1)$ has at least two upper neighbors, and all elements u of U satisfy $a(1) \geq u \geq b(1)$.



If $A = k\bar{\Delta}/I$, where $\bar{\Delta}$ is a quiver formed by attaching one end of a quiver C of type \mathbb{A}_r to a quiver Δ' , and the admissible ideal I is generated by elements in $k\Delta'$, then the Tits form for A is independent of the orientation of C . Algebras obtained from a given algebra A by changing the orientation for various subquivers C in this fashion are said to be obtained

from A by a *reorientation of arms*. Thus we have shown the following corollary to the theorem.

Corollary. *If $A = kS$ is a representation-directed algebra with a sincere diamond, and if $A' = kS'$ is obtained from A by a reorientation of arms, then A' is a representation-directed algebra such that $\varepsilon = (1, \dots, 1)$ is the only sincere positive 1-root of the Tits form of A' .*

3. The combinatorial part of the proof

Lemma 1. *If $A = kS/J$ is a representation-directed algebra, and if $\varepsilon = (1, \dots, 1)$ is the only sincere positive 1-root of q_A , then $J = 0$.*

Proof. If X is the indecomposable A -module with $\mathbf{dim} X = \varepsilon$, then by [5] we know that $X \cong \delta_{\text{supp } X} = \delta_S$. Hence S is relation-free, and therefore $J = 0$. \square

Lemma 2. *Suppose that $A = kS$ is a representation-directed algebra, that $\varepsilon = (1, \dots, 1)$ is the only sincere positive 1-root of q_A , and that x is an element of S which is neither minimal nor maximal. If S' is the full subposet of S associated with $S \setminus \{x\}$, then $A' = kS'$ is representation-directed, and $\varepsilon' = (1, \dots, 1)$ is the only positive sincere 1-root of $q_{A'}$.*

Proof. It is clear that S' is connected, kS' is representation-directed, and $X' = \delta_{S'}$ is a sincere indecomposable kS' -module. We assume that there is another sincere indecomposable kS' -module Y' , different from X' . From [5], there has to be an element y of S' , satisfying $\dim_k Y'(y) \geq 2$. Let L be the left adjoint of the restriction functor from the category of kS -modules to the category of kS' -modules. Hence LY' is an indecomposable kS -module such that $LY'(z) = Y'(z)$ for all elements z of S different from x . From [2], the support of LY' is convex, and therefore LY' is a sincere module, not isomorphic to δ_S , a contradiction. \square

Before continuing, we have another prerequisite. Let $(-, -)_A$ be the symmetric bilinear form associated with the quadratic form q_A , and let σ_i be the reflection with respect to $(-, -)_A$ along the canonical base vector $e(i)$ for $i = 1, \dots, r$. This means that $\sigma_i(x) = x - 2(e(i), x)_A e(i)$ for all x in \mathbb{Z}^r . For x a 1-root, the vector $\sigma_i(x)$ is also a 1-root of q_A . In particular, if ε is the only sincere positive 1-root of q_A , then $2(e(i), \varepsilon)_A \geq 0$ for all $i = 1, \dots, r$, because otherwise $\sigma_i(\varepsilon)$ would be another sincere positive 1-root.

Proposition 2. *If $A = kS$ is a representation-directed algebra such that $\varepsilon = (1, \dots, 1)$ is the only positive 1-root of the Tits form q_A , then A is obtained from a representation-directed algebra with a sincere diamond by a reorientation of arms.*

Proof. We proceed by induction on the cardinality r of S , and observe that for $r = 1$ there is nothing to prove. For $r > 1$ we first consider the case where every element of S is either maximal or minimal. Thus $kS = k\bar{\Delta}$ is a hereditary algebra of finite representation type. By Gabriel's theorem (see [7]) the graph Δ underlying $\bar{\Delta}$ has to be one of the Dynkin diagrams $\mathbb{A}_r, \mathbb{D}_r$ or $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$. But for all of these diagrams except \mathbb{A}_r , the corresponding Tits forms have more than one sincere positive 1-root.

Now we have to deal with the case where there exists an element x of S which is neither minimal nor maximal. By Lemma 2, we can apply induction to the full subposet S' of S associated with $S \setminus \{x\}$. Let us consider the Hasse diagram $\bar{\Delta}'$ of S' . The case where $\bar{\Delta}'$ is a graph of type \mathbb{A}_{r-1} is clear. Otherwise, $\bar{\Delta}'$ has the shape given in Figure 1.

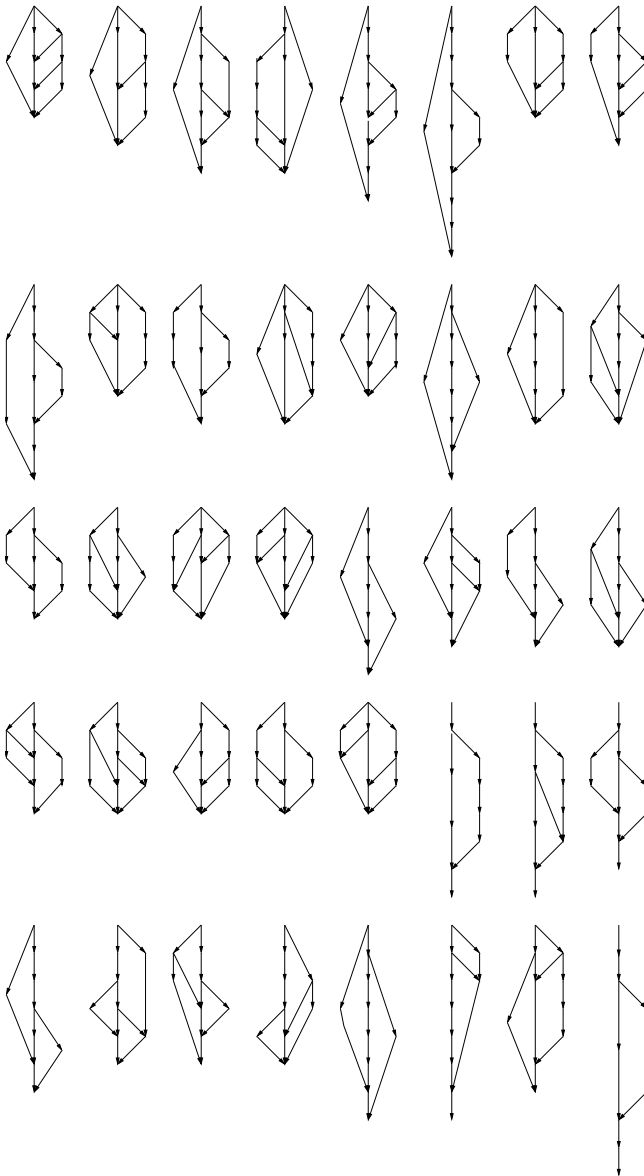


Figure 2: Hasse diagrams for the maximal exceptional representation-directed algebras having a faithful diamond (Continued in Figure 3)

Representation-directed diamonds

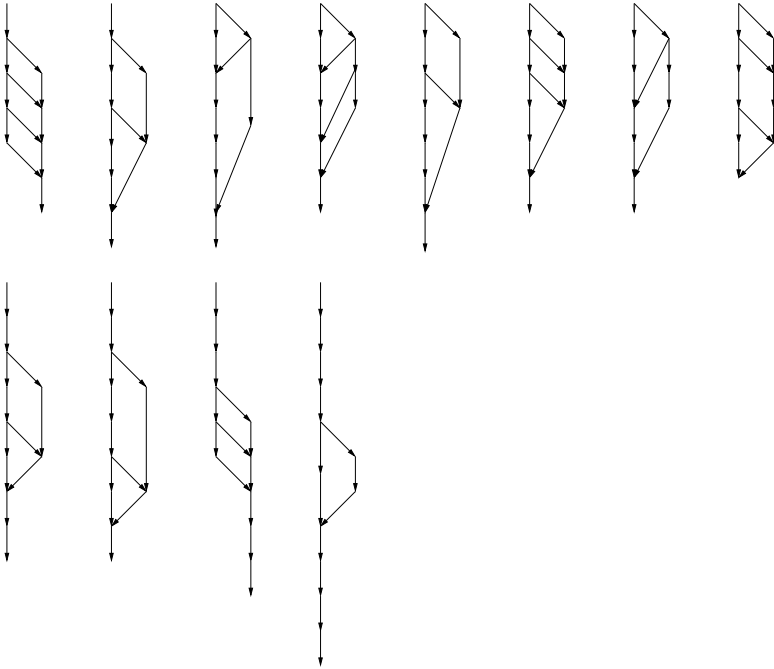


Figure 3: Hasse diagrams for the maximal exceptional representation-directed algebras having a faithful diamond (Continued from Figure 2)

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