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COSET ENUMERATION USING PREFIX GRÖBNER BASES: AN EXPERIMENTAL APPROACH

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Abstract

We study a new method for coset enumeration in finitely presented groups. Our method uses prefix Gröbner basis computation in the monoid ring $\mathbb{K}[\mathcal{M}]$, where $\mathbb{K}$ is a computable field and $\mathcal{M}$ a monoid presented by a convergent string-rewriting system. The method is compared to well-known methods for Todd–Coxeter enumeration, using examples from the literature where studies of these methods are reported. New insights into coset enumeration were gained using three different kinds of orderings, combined with new frameworks and strategies implemented in MRC 1.2.

1. *Introduction*

In 1936, Todd and Coxeter [29] developed a procedure to enumerate the cosets of a subgroup of a group; this turned out to be a strong tool for studying finitely presented groups in combinatorial group theory. The first computer implementation was that of Haselgrove, in 1953. Leech described this method, together with other early implementations, in [15]. Various different approaches to coset enumeration can be found in [5, 8, 16, 20, 27, 28], for example. These include other approaches to coset enumeration, using the Knuth–Bendix procedure or automata. Nowadays, coset enumeration is implemented in computer algebra systems for groups, such as GAP [7], Magma [1], and ACE2 [9, 21]. Implementations of the Knuth–Bendix procedure are provided by KBMAG [14], XSSR [13], and Waldmeister [2, 3, 4, 12].

Coset enumeration takes as its input a finitely presented group and a finitely presented subgroup. It then attempts to find the index of the subgroup in the whole group. If this index is finite, the procedure will succeed. However, the enumeration process proves to be difficult for various reasons, the major one being the absence of an upper bound for the number of cosets to be enumerated. This is due to the undecidability of the underlying problem. Further, some approaches that perform well for one such enumeration might perform very badly for another one, compared to other approaches. Finally, the performance of coset enumeration depends in general on the presentation of the group. For example, the Macdonald groups $G(3, 3)$, $G(3, -1)$, and $G(-1, -1)$ are all isomorphic, but while coset enumeration is trivial for the latter two presentations, it is much more difficult for the first one. Other examples show the same behaviour.

A description of a procedure simulating the Todd–Coxeter procedure using prefix Gröbner bases (see [19], for example), and a comparison with Knuth–Bendix completion, were presented in [22, 23]. There, the relationship between string-rewriting methods and Todd–Coxeter enumeration was described, whereas in this paper, we describe how the procedure

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presented in [23] can be improved, with real computation in mind. In order to do so, we introduce so-called 'frameworks', which can be compared to the Felsch and HLT methods for the original Todd–Coxeter enumeration procedure. They are supplemented by clever strategies for adding potential cosets.

Our method makes it possible to study coset enumertion from yet another point of view. All the other methods currently in use seem to use length-lexicographical orderings, with the precedence of the letters being the only parameter, but our method supports all types of orderings, provided that they are well founded. Three different types of ordering (Knuth–Bendix, syllable and, of course, length-lexicographical orderings) have already been implemented in MRC 1.2 (see [24, 25, 30]). We have studied their influence on the enumeration process, and have examined how different combinations of frameworks, strategies and orderings behave. Results achieved using the Knuth–Bendix and syllable orderings are sometimes much better, compared with the length-lexicographical ones.

Another new aspect that has been studied is coset enumeration over general groups, instead of the free group that is normally used. (Let a free group be given by a set of generators and a set of rules. Then a group is defined by a set of regulators modulo this free group. We can move some of the relators defining the group from the set of relators to the set of rules of the underlying free group. Now the underlying group is no longer a free group, and has to be completed using the Knuth–Bendix procedure. This allows even more variations.)

As the prefix Gröbner basis setting in MRC 1.2 produces a lot of overhead, we considered the numbers of cosets enumerated, rather than measuring the processing time taken. In reality, a coset-enumeration procedure based on prefix string rewriting would be sufficient for the examples considered here. On the other hand, the examples presented by Linton in [17] can be computed using the coset-enumeration procedure based on prefix Gröbner bases, but not with the procedure restricted to prefix string rewriting. This justified an extensive study of coset enumeration based on prefix Gröbner bases, and not only on prefix string rewriting. However, if our findings prove to be valuable, they could perhaps be incorporated into existing implementations of Todd–Coxeter enumeration procedures; if this is not possible, a specialized reimplementation on the basis of prefix string rewriting might be feasible.

In [5, 8] experiments are described to assess the performance of various styles of Todd–Coxeter enumeration procedures, namely Felsch, HLT, and Lookahead-HLT. In addition, more recent results can be found in [10, 11]. We present the results that were achieved using our methods for the examples presented in [5, 8]. As expected, our method is sometimes better, and sometimes worse, than the existing methods like Felsch- or HLT-style enumeration procedures. Interestingly, the results for our method are considerably better for some of the examples.

Further, if we compare our approach to the findings of Havas presented in [8], we get the following results. First of all, all the methods (including our own) are syntactical ones. While the existing methods use ingenious strategies to fill the coset table, our strategies introduce new cosets based on the multiplication table represented by the prefix Gröbner basis. We found that adding the symmetric relators considerably improved the enumeration process; this was already known for the other methods. Also, our method treats all the group relators as subgroup generators per se; this technique was introduced for Felsch-style enumeration by Havas in [8].

The paper is organized as follows.

In Section 2, we recall the most important concepts from coset enumeration. Next, we review the Felsch- and HLT-style enumeration procedures. In addition, we present the procedure for simulating the Todd–Coxeter enumeration based on prefix Gröbner bases, and the concept of coset enumeration over general groups.

As the simulation procedure is appropriate only for small examples, we propose two frameworks and various strategies that improve the performance of the enumeration. These are presented in Section 3, together with a comparison to Felsch- and HLT-style enumeration. As orderings play an important role, these are introduced in Section 4. In [**5**], pathological and non-pathological examples are distinguished. The non-pathological examples and the results obtained with our method are presented in Section 5. The pathological examples used for this paper are taken from [**5, 8**], and are presented in Section 6. The frameworks and strategies were evaluated, and the result of this evaluation is described in Section 7. In Section 8 the influence of the ordering on the coset enumeration is described for the pathological cases. The results obtained for a closer analysis of the Macdonald groups (see also [**18**]) are described in Section 9. In Section 10 the examples for coset enumeration over general groups are presented, together with the results obtained for them. Possible enhancements are pointed out in Section 11, and the results are tabulated in Appendix A. A first version of this paper was published as a technical report in [**26**].

2. *Theoretical foundations*

In this section, the theoretical foundations of coset enumeration are presented. In Section 2.1, the general idea of coset enumeration is described, together with the famous procedure of Todd and Coxeter. Next, three coset-enumeration procedures based on the original one are described, namely Felsch-, HLT-, and Lookahead-HLT-style procedures. In Section 2.2, a procedure for the simulation of Todd–Coxeter coset enumeration based on prefix Gröbner bases of right ideals in free group rings is described. This was first presented in [**23**]. A generalization of this procedure, based on prefix Gröbner bases of right ideals in group rings, is presented in Section 2.3.

2.1. *Todd–Coxeter enumeration*

Todd–Coxeter coset enumeration is a famous method used in combinatorial group theory for studying finitely presented groups. It is based on the following fundamental observations. Presenting a group $\mathcal{G}$ in terms of generators X and relators R corresponds to viewing it as the quotient of the free group $\mathcal{F}$ (generated by X) by the normal subgroup $\mathcal{N}$ generated by R. $\mathcal{N}$ can be viewed as the subgroup of $\mathcal{F}$ generated by $N(R) = \{w \circ r \circ w^{-1} \mid w \in \mathcal{F}, r \in R\}$. Notice that if R is finite, then $\mathcal{N}$, while finitely generated as a normal subgroup of $\mathcal{F}$, need not be finitely generated as a subgroup.

Now, given a subgroup $\mathcal{S}$ of $\mathcal{G}$ for $g \in \mathcal{G}$, we can study the *cosets* $\mathcal{S}g = \{s \circ g \mid s \in \mathcal{S}\}$ of $\mathcal{S}$ in $\mathcal{G}$. Since for $g, h \in \mathcal{G}$, either $\mathcal{S}g = \mathcal{S}h$ or $\mathcal{S}g \cap \mathcal{S}h = \emptyset$, the group $\mathcal{G}$ is a disjoint union of cosets, and the number of different cosets is called the *index* $|\mathcal{G} : \mathcal{S}|$ *of* $\mathcal{S}$ *in* $\mathcal{G}$. We know that if $\mathcal{S}$ is generated by a set $S \subseteq \mathcal{G}$, the index of $\mathcal{S}$ in $\mathcal{G}$ is the same as the index of the subgroup $\mathcal{H}$ generated by $S \cup N(R)$ in $\mathcal{F}$. While it is undecidable whether a particular subgroup has a finite index in a group, coset enumeration attempts to *verify* whether the index is finite by enumerating the (potential) cosets and their multiplication table.

Three different procedures for performing Todd–Coxeter enumeration will be summarized next. Detailed descriptions of the procedures can be found in [29, 5, 20, 8, 28], for example. All these enumeration procedures essentially use a table that describes the result of multiplying each coset with a generator of the group. The rows are labelled with the cosets enumerated thus far, while the columns are labelled with the generators of the group. The goal of all the procedures is to completely fill this table. If the index is finite, the procedure terminates with a complete multiplication table for the cosets; in this paper, we suppose that we do have a finite index. The procedures differ, however, in how the coset table is filled. Two steps are essential for filling this table: the definition of new cosets, and the extraction of information from the relators defining the group and the subgroup, respectively. Each of the procedures applies these steps in a particular order.

Felsch-style procedures first extract as much information as possible from the relators. If no more information is available, a new coset is defined, and a row is added for this coset. This process is repeated until the table is completely filled.

On the other hand, Haselgrove–Leech–Trotter (HLT)-style procedures define as many new cosets as are necessary to complete a row of the multiplication table. This might imply a necessity to introduce new cosets, which leads to the generation of new rows in the multiplication table.While Felsch-style procedures complete the table by defining only as many cosets as are necessary to complete a row (and thus, in general, fewer than HLT-style procedures do), they are less efficient than the latter, as the extraction of information is costly. On the other hand, HLT-style procedures define many more cosets than Felsch-style procedures do, but are normally faster.

A third method, introduced in [5], is called ‘Lookahead-HLT’, and tries to combine the advantages of both HLT- and Felsch-style procedures. It consists of two phases: a defining phase, and a lookahead phase. First, the procedure proceeds like HLT, and defines new cosets in order to close rows. When a certain limit on the number of cosets (rows), L, is reached, the defining phase is stopped, and the lookahead phase is entered. The lookahead phase extracts as much information as possible from the relators. If collapses occur (that is, if a duplication of cosets is discovered), then the number of cosets decreases, and hence becomes less than L. If the table is not complete after this phase, but the number of cosets is less than L, then the lookahead phase is terminated, and the defining phase is entered again. This cycle is repeated until either the table is complete, or no more cosets can be defined and no more information can be extracted from the relators. In the latter case, the enumeration has to be repeated with a larger bound L. In [8, 10, 11], different strategies are described that may improve the enumeration in certain cases.

In order to assess the performance of the coset-enumeration procedures, different measures were described in [5]. One of them is of course the time needed to compute a certain example on a certain machine. Two other important measures are the maximal number and the total number of cosets defined. The *maximal number of cosets* is the maximum number of cosets stored in the coset table at any one time. The *total number of cosets* is the number of cosets generated during the enumeration process. These measures can be related to the time and space required by the respective procedures, as the maximal number is equivalent to the maximal size of the table needed by the enumeration procedure, while the total number is equivalent to the duration of the enumeration. The other measures used in the assessment were essentially based on these three measures.

The advantage of using the maximal and total numbers of cosets defined as the measures for comparing coset-enumeration procedures is that they are independent of the computers on which the procedures are executed.

Procedure: extended_todd_coxeter_simulation

Given: $F_R = \{r - 1 \mid r \in R\} \neq \emptyset$, a set of binomials representing the relators.
$F_S = \{s - 1 \mid s \in S\}$, a set of binomials representing the subgroup generators.
Find: N: the cosets of the subgroup S in the group G
G: the prefix Gröbner basis representing a multiplication table

$N := \{\lambda\}$;
$B := \{a \mid a \in X \cup X^{-1}\}$;
$G :=$ prefix_groebner_basis_of_right_ideal_fg$(F_R \cup F_S)$;
while $B \neq \emptyset$ **do**
  $\tau :=$ min$_<(B)$;
  $B := B \backslash \{\tau\}$;
  if τ is not prefix reducible by G
    then $N := N \cup \{\tau\}$;
      $B := B \cup \{\tau a \mid a \in (X \cup X^{-1}) \backslash \{(\text{last}(\tau))^{-1}\}\}$;
      % where last(τ) is the last letter of the word τ
      $H := \{\tau * (r - 1) \mid r - 1 \in F_R\}$;
      $G :=$ prefix_groebner_basis_of_right_ideal_fg$(G \cup H)$;
      $N := N \backslash \{w \in N \mid w$ is prefix reducible by $G\}$;
  endif
endwhile

Procedure 1: Extended Todd–Coxeter simulation

2.2. *Todd–Coxeter simulation*

In [23], Procedure 1 is presented; this simulates the Todd–Coxeter procedure to enumerate cosets using the computation of prefix Gröbner bases in free group rings (see [25]).

Let the free group $\mathcal{F}$ be generated by X. Let $\Sigma = X \cup X^{-1}$, where $X^{-1} = \{x^{-1} \mid x \in X\}$. A word w is an element of Σ^*. Besides the sets F_R and F_S, which contain the encoded relators and the subgroup relators respectively, we have three sets.

First of all, N is a set of words, and contains the potential coset representatives of the subgroup generated by $S \cup N(R)$ in $\mathcal{F}$. If the procedure terminates, then N ultimately contains the coset representatives. Next, there is G, a prefix Gröbner basis, which represents the multiplication table obtained thus far, and which is used to decide whether or not elements of N are indeed coset representatives. In fact, only those words that are *not* prefix-reducible with an element of G are potential cosets. On termination, all the prefixes of the head terms of the polynomials of G form the set of coset representatives; that is, the latter set is equal to $N = \{w \mid w$ is prefix of $\mathsf{HT}(p),\ p \in G\}$. The border set $B \subseteq \mathcal{F}$ also contains words, and serves as a test set for possible coset representatives.

The set N is initialized as the set containing only the empty word, where the empty word is the coset representative of the subgroup itself. During the computation, this set remains *prefix closed*; that is, for every word w in Σ^* such that w is in N, all prefixes v in Σ^* for which there exists some u in Σ^* such that vu is equivalent to w, are in N, too. The border set B is initialized with all the elements of the alphabet Σ. Finally, the set G is computed as the interreduced prefix Gröbner basis of the union of the group relators and the subgroup generators, $F_R \cup F_S$.

Now, the procedure proceeds as follows. As long as there are elements in the border set B, the smallest element of B is chosen and removed from B. A length-lexicographical ordering on the elements is assumed. If this element is not prefix-reducible with any of the elements of G, then we have a potential coset representative. It is added to the set of potential coset representatives, N. Further, the border set B is expanded by adding all the multiples of this element with the letters of the alphabet Σ, except for the inverse of the last letter of this element. (The latter word has already been examined.) All the multiples of the potential coset representative with the relators F_R are added to G. Now, the multiplication table G is recomputed. Finally, all elements from N that are now prefix-reducible with an element of G are removed, as they are no longer potential coset representatives. If the element selected from B is prefix-reducible with one of the elements of G, then it is not a coset representative. As all multiples of this element are prefix-reducible with the same element from G, they are also not added to B. Thus the element is simply removed from B.

We get the following two theorems.

Theorem 1. *The procedure* extended_todd_coxeter_simulation *terminates if and only if the subgroup generated by $S \cup N(R)$ has finite index in $\mathcal{F}$.*

Theorem 2. *If the procedure* extended_todd_coxeter_simulation *terminates, then N contains the coset representatives $\mathcal{S}g$, where $g \in \mathcal{G}$.*

The set N of potential cosets is important for the evaluation of the performance of the procedure. For Felsch, HLT, and Lookahead-HLT, the maximal and total numbers of cosets enumerated were computed, and were used to assess the performance of each procedure. As N contains the potential cosets at any given time during the computation, we see that the maximal number of cosets is the maximal number of N at any time during the computation. Further, the number of elements added to N during the computation is the total number of cosets enumerated. In the sequel, we shall write 'n/m', where n is the maximal and m the total number of cosets defined.

2.3. *Coset enumeration over general groups: theory*

While the original approach is based on the idea of computing the index of the subgroup $\mathcal{H}$ generated by $S \cup N(R)$ in the free group $\mathcal{F}$, allowing the use of prefix Gröbner basis techniques in $\mathbb{K}[\mathcal{F}]$, a splitting of the set of relators R can lead to new enumeration procedures. For $R = R_1 \cup R_2$, where R_1 is complete as a set of rules, let $\mathcal{G}_1$ be the group presented by $\langle \Sigma, R_1 \rangle$. Alternatively, if R_1 can be finitely completed to $\widetilde{R_1}$, let $\mathcal{G}_1$ be the group presented by $\langle \Sigma, \widetilde{R_1} \rangle$. Then we can try to compute, in a similar fashion, the index of the subgroup $\mathcal{H}_1$ generated by $S \cup N(R_2)$ in $\mathcal{G}_1$; this is, of course, again $|\mathcal{G}_1 : \mathcal{H}_1| = |\mathcal{F} : \mathcal{H}| = |\mathcal{G} : \mathcal{S}|$. The resulting procedure is nearly identical to the original extended_todd_coxeter_simulation procedure on inputs F_{R_2} and F_S, except that now the computation takes place in $\mathbb{K}[\mathcal{G}_1]$, and we have to be more careful in choosing τ to ensure fairness. Further, the prefix_groebner_basis_of_right_ideal_fg procedure, computing prefix Gröbner bases in free group rings, has to be replaced by the reduced_prefix_groebner_basis_of_right_ideal procedure, which computes prefix Gröbner bases in arbitrary group rings (see [25]).

Let the free group be given as $\mathcal{F} = (X, T)$, where

$$T = \{(aa^{-1}, \lambda) \mid a \in X\} \cup \{(a^{-1}a, \lambda) \mid a \in X\}.$$

Let R be a set of relators. If one of the relators has the form $r = \sigma^n$, for $\sigma \in \Sigma$ and $n \in \mathbb{N}^+$, then we can remove this relator from R and add it to T. The latter is then completed with respect to the ordering required, using the Knuth–Bendix completion procedure, which in this case is always terminating. We thus get T', the completed set of rules, and $R' = R\backslash\{r\}$. Using these, we can perform coset enumeration as before. Other relators, too, can be moved, provided that a finite and convergent presentation exists for the resulting group.

3. *Frameworks and strategies*

The original extended_todd_coxeter_simulation procedure was implemented in MRC 1.2 (see also [**25**]), and examples known from the literature (see [**5, 8**]) were computed. It soon became clear that the original version is far too slow to be of any use, except for very small examples. This is not surprising, as the prefix Gröbner basis computation that is performed after each new coset element is added is quite costly (see also Section 3.3 for a comparison to known methods for coset enumeration). As a consequence, it is necessary to define a certain number of cosets *before* recomputing the multiplication table, that is, the prefix Gröbner basis. This might lead to the introduction of a large number of unnecessary cosets, but for larger examples this is much faster than the simple approach, as fewer prefix Gröbner basis computations are necessary. Now, the set G need not always be a prefix Gröbner basis, but is still a right ideal basis.

Various frameworks and strategies have therefore been developed in order to find out whether the coset-enumeration process described in this paper is useful when compared to currently known procedures and strategies. For implementational reasons, the time and space requirements are much higher for our implementation, compared to already available implementations of the Todd–Coxeter procedure. First of all, in MRC 1.2, cosets are represented by words, and not by numbers; this requires more space for large examples. Second, the coefficients are elements of $\mathbb{Q}$, even though they can take only the values $\{-1, 0, 1\}$. Third, since MRC 1.2 handles polynomials, we are encoding the relators as polynomials, where the use of binomials would be sufficient. Customizing the procedure to these settings drastically reduced the time and space requirements. These customizations can be done only for the examples considered here, however, and not, for example, for those of [**17**], where polynomials are needed.

However, it was more important to us to discover whether our system can compete with respect to the crucial numbers in Todd–Coxeter enumeration: the maximal and total numbers of cosets defined during the enumeration process. Since the enumeration is performed in a different algebraic structure, new parameters would play an important role, and might help us to study cosets from a different point of view.

Two different frameworks have been considered, and are described in Sections 3.1 and 3.2. They are compared to two methods for Todd–Coxeter enumeration in Section 3.3. The frameworks use two kinds of procedures. First of all, there is a procedure to compute the respective prefix Gröbner basis. This might be either the general one in the case of general groups, or a special one in the case of free groups (see [**25**]). In the procedures presented below for the different frameworks, the latter was used. For the case of general groups, it would have to be replaced by the general one.

Further, the frameworks use the additional_elements_start and additional_elements procedures. These procedures add to the border set (after the first prefix Gröbner basis computation, and after each subsequent such computation, respectively) elements that would not normally be considered until much later in the enumeration process. These will be described in Section 3.4.

Both frameworks test to see whether the element τ that is selected and removed from B is a new coset. This test includes checking whether τ is prefix-reducible by any element of G, as before. In addition, checks are run to see whether τ is already in N. (This may happen, as the strategies may generate elements that are already in N.) It should be noted that the procedures as described above are simplified versions of those that were actually implemented.

This was done in order to present them more clearly, and to omit design details that have no influence on the functionality.

Both frameworks depend on the organization of the border set. This is currently organized as a queue, with an additional data structure ordered set that is used to check whether elements are already in the border set. The queue implies that it is always the first element of the queue that is selected, and that new elements are added to the end. As the characters of the alphabet are added to the border set in the same order as that in which they are read from the input, the elements are considered from 'smallest' to 'largest' with respect to a length-lexicographical ordering, where the first letter read is considered as 'small', and the last one as 'large'.

3.1. *The 'steps' framework*

The first framework considered uses a value called *recompute*, provided by the user. After this number of new elements have been added to the coset table N, the prefix Gröbner basis computation is performed. Procedure 2 (see page 82) implements this framework.

The differences from the original procedure are as follows. The most important difference is the value *recompute*, which triggers the system to recompute the prefix Gröbner basis. Every time a number of cosets equal to the value of *recompute* have been added to N, the prefix Gröbner basis of G is computed, and the set of potential cosets N is adapted accordingly. As G might not yet be a prefix Gröbner basis on termination of the **while**-loop, the last four lines of the procedure are needed. Whenever the counter is not equal to zero on termination of the **while**-loop, we have to compute the prefix Gröbner basis of G and recompute the coset set N. The only other changes are the additions due to the strategies being used; that is, the addition of elements to B at the beginning and after each prefix Gröbner basis computation. The select procedure, which selects one element of the border set, has to use a fair strategy to do so.

Unfortunately, the choice of the *recompute* parameter is not an obvious one, as it is not known in advance whether or not the number of cosets is finite. In particular, it is not known how many cosets will be generated. There are even examples where the number of cosets that have to be defined exceeds the number of cosets in the final result. Thus, a reasonably good value can be chosen only if the result (that is, the total and maximal numbers of cosets) is already known before the computation is done; in general, however, this is impossible for unknown examples at the moment.

This framework depends on the order in which elements of the same length are considered. Changing the order in which the characters are given to the program, or the way in which elements are selected from the border set, immediately changes the results, unless $\textit{recompute} = 1$.

Procedure: extended_todd_coxeter_simulation_steps

Given: $F_R = \{r - 1 \mid r \in R\} \neq \emptyset$, a set of binomials representing the relators.
$F_S = \{s - 1 \mid s \in S\}$, a set of binomials representing the subgroup generators.
recompute, value to trigger computation of the multiplication table.

Find: N: the cosets of the subgroup S in the group G
G: the prefix Gröbner basis representing a multiplication table

```
N := {λ};
G := prefix_groebner_basis_of_right_ideal_fg(F_R ∪ F_S);
B := {a | a ∈ X ∪ X^{-1}};
B := B ∪ additional_elements_start(G, R, N, B);
counter_t := 0;
while B ≠ ∅ do
    τ := select(B);
    B := B\{τ};
    if τ is a new coset
        then  counter_t := counter_t + 1;
              N := N ∪ {τ};
              B := B ∪ {τa | a ∈ (X ∪ X^{-1})\{(last(τ))^{-1}}};
              G := G ∪ {τ * (r − 1) | r − 1 ∈ F_R};
              if counter_t ⩾ recompute
                  then  counter_t := 0;
                        G := prefix_groebner_basis_of_right_ideal_fg(G);
                        N := N\{w ∈ N | w is prefix reducible by G};
                        B := B ∪ additional_elements(G, R, N, B);
              endif
    endif
endwhile
if counter_t ≠ 0
    then  G := prefix_groebner_basis_of_right_ideal_fg(G);
          N := N\{w ∈ N | w is prefix reducible by G};
endif
```

Procedure 2: Extended Todd–Coxeter simulation: steps framework

3.2. *The 'level' framework*

The 'level' framework that is implemented by Procedure 3 (see page 83) is independent of such a value. Instead of a fixed number supplied by the user, an internal condition triggers the prefix Gröbner basis computation.

At the beginning, only the letters of the alphabet are added to the border set B. This is called *Level 1*. Each element that is selected from the border set B, and that is also a new coset, is added to the set of cosets, N. Further, this element is multiplied by all the letters of the alphabet, except for the inverse of its last letter. These elements have length 2, and are added to the new border set, B_{new}. Finally, the multiplication table G is extended. If the current border set is empty, the new border set becomes current, and the prefix Gröbner

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Procedure: extended_todd_coxeter_simulation_level

Given: $F_R = \{r - 1 \mid r \in R\} \neq \emptyset$, a set of binomials representing the relators.
$F_S = \{s - 1 \mid s \in S\}$, a set of binomials representing the subgroup generators.
Find: N: the cosets of the subgroup S in the group G
G: the prefix Gröbner basis representing a multiplication table

$N := \{\lambda\}$;
$G :=$ prefix_groebner_basis_of_right_ideal_fg$(F_R \cup F_S)$;
$B := \{a \mid a \in X \cup X^{-1}\}$;
$B := B \cup$ additional_elements_start(G, R, N, B);
$B_{\text{new}} := \emptyset$;
while $B \neq \emptyset$ **do**
  $\tau :=$ select(B);
  $B := B \backslash \{\tau\}$;
  if τ is a new coset
    then $N := N \cup \{\tau\}$;
      $B_{\text{new}} := B_{\text{new}} \cup \{\tau a \mid a \in (X \cup X^{-1}) \backslash \{(\text{last}(\tau))^{-1}\}\}$;
      $G := G \cup \{\tau * (r - 1) \mid r - 1 \in F_R\}$;
  endif
  if $B = \emptyset$
    then $B := B_{\text{new}}$;
      $B_{\text{new}} := \emptyset$;
      $G :=$ prefix_groebner_basis_of_right_ideal_fg(G);
      $N := N \backslash \{w \in N \mid w$ is prefix reducible by $G\}$;
      if $B \neq \emptyset$
        then $B := B \cup$ additional_elements(G, R, N, B);
      endif
  endif
endwhile

Procedure 3: Extended Todd–Coxeter simulation: level framework

basis computation is performed. Thus, a border set of Level n consists of all the elements of length n that were generated by the previous level, $n-1$. These are examined before the prefix Gröbner basis of G is computed again. As the recomputation of the prefix Gröbner basis is triggered by the procedure itself, this framework is self-adapting with respect to the number of new cosets defined between any two computations of the prefix Gröbner basis.

As all the elements of the current level n (that is, all elements of length n) are considered before the recomputation of G, this method is independent of the order in which these elements are considered. Nevertheless, this framework, still depends on the method that is used to select elements from the border set. Of course, the properties concerning the length of the elements hold only if no additional elements are added by the strategies.

Several variations of this framework were examined. All the variants attempt to compute the prefix Gröbner basis more often, although perhaps only partially.

The first variation alternately considers the current border set and the additional elements, before computing the prefix Gröbner basis. The second variation consists of computing the prefix Gröbner basis after the first half of the current border set has been examined, and then again after the second half. The third variation uses an incremental approach. One reduction cycle of the prefix Gröbner basis computation is performed each time a certain number of elements have been added to the right ideal basis G. There exist several possibilities for additional restrictions. The computation can be performed

1. after each element added to G whose normal form is smaller than the product of coset and relator, or
2. after each element added to G whose normal form is smaller than the current level.

This can be further restricted to every nth element with these properties, where n could be, for example, $|N|/100$.

The first two variations sometimes produce better results, but do not do so in general. They were not examined in detail. The last variant produces better results, but is much more costly. Even the partial computation of the prefix Gröbner basis is very costly, and the results improve only if this computation is performed more often.

3.3. *Comparison with Todd–Coxeter methods*

In [**5, 8**] three methods for Todd–Coxeter enumeration were described in detail (see also Section 2.1). Felsch-style methods try to extract as much information as possible from the relators and the current coset table, and define new cosets only if necessary. HLT and Lookahead-HLT (which will be referred to as 'HLT' below) define as many new cosets as possible, and extract new information from the relators and the coset table only if necessary.

If we compare our frameworks to these methods, it can be seen that the original procedure follows the Felsch-style methods, and has the same drawbacks: the extraction of information is costly. The 'steps' framework and the 'level' framework are both similar to HLT: they define a certain number of new cosets before extracting new information. The difference is that HLT uses some fixed-size table (that is, the maximal number of definable cosets is fixed beforehand), while our frameworks limit the number of newly defined cosets, and the coset table (that is, the ideal basis) is allowed to grow.

While the 'steps' framework has a similar problem to that of HLT (that is, the number of cosets to be defined before extracting new information must be given before starting the computation), this is not the case for the 'level' framework.

3.4. *Strategies*

Already known coset-enumeration procedures try to use 'clever' strategies to select the coset to be considered next. These strategies are based on information about the multiplication table.

The two frameworks would normally select the cosets in increasing order, beginning with the smallest. However, there might be cosets that are quite large with respect to the ordering, but which lead to important information (for example, that two or more cosets are identical). The enumeration process might be considerably shortened by using this information. Thus, in order to enhance the frameworks, strategies were provided to add elements to the border set that would normally be considered only much later in the enumeration process.

The procedures for implementing the strategies take four parameters. The first two parameters (namely the right ideal basis G, and the set of relators R) are used to determine which potential elements should be added, according to the strategy being used. The two other parameters (namely the coset set N and the border set B) are used to test whether the potential elements have already been considered. Elements are added only if they are neither already in the coset set N, nor in the border set B, and if they are not prefix-reducible using polynomials from G. This reduces the maximal size of both the border set B and the new border set B_{new}.

It is not obvious which elements should be considered, nor when they should be added. Elements are currently added using the additional_elements_start and additional_elements procedures, which add elements to the border set after the first and each subsequent prefix Gröbner basis computation, respectively, according to the following strategies.

1. NONE: No elements are added.
2. P-ALL: additional_elements_start and additional_elements add all the prefixes of the head-terms of all the polynomials of G:
$$\{w \mid w \in \Sigma^+ \text{ and } \exists p \in G\ \exists v \in \Sigma^* : wv \equiv \mathsf{HT}(p)\}.$$
3. P-R: additional_elements_start and additional_elements add those prefixes of the head-terms of all the polynomials of G which, when multiplied with the generators, could lead to polynomials which can be reduced using G:
$$\{u \mid u \in \Sigma^+ \text{ and } \exists w \in \Sigma^+ : [\exists p \in G\ \exists v \in \Sigma^* : wv \equiv \mathsf{HT}(p) \text{ and } \exists v \in \Sigma^+\ \exists z \in \Sigma^*\ \exists r \in R : uv \equiv w \wedge vz \equiv \mathsf{HT}(r)]\}.$$
4. P-G: additional_elements_start and additional_elements add those prefixes of the head-terms of all the polynomials of G which, when multiplied with polynomials of G, could lead to polynomials which can be reduced using G:
$$\{u \mid u \in \Sigma^+ \text{ and } \exists w \in \Sigma^+ : [\exists p \in G\ \exists v \in \Sigma^* : wv \equiv \mathsf{HT}(p) \text{ and } \exists v \in \Sigma^+\ \exists z \in \Sigma^*\ \exists g \in G : uv \equiv w \wedge vz \equiv \mathsf{HT}(g)]\}$$
5. I-ALL: additional_elements_start and additional_elements add all the inverse terms of the prefixes of the head-terms of all the polynomials of G:
$$\{w^{-1} \mid w \in \Sigma^+ \text{ and } \exists p \in G\ \exists v \in \Sigma^* : wv \equiv \mathsf{HT}(p)\}.$$
6. I-R: additional_elements_start adds all the inverse terms of the head-terms of the relators:
$$\{w^{-1} \mid w \in \Sigma^+ \text{ and } \exists r \in R\ \exists v \in \Sigma^* : wv \equiv \mathsf{HT}(r)\}.$$
additional_elements adds no elements to the border set.
7. I-R-P: additional_elements_start adds all the inverse terms of the head-terms of the relators:
$$\{w^{-1} \mid w \in \Sigma^+ \text{ and } \exists r \in R\ \exists v \in \Sigma^* : wv \equiv \mathsf{HT}(r)\}.$$
additional_elements_start and additional_elements add all the prefixes of the head-terms of all the polynomials of G:
$$\{w \mid w \in \Sigma^+ \text{ and } \exists p \in G\ \exists v \in \Sigma^* : wv \equiv \mathsf{HT}(p)\}.$$
8. ENUM: additional_elements_start and additional_elements add elements in a special order.
9. RANDOM: additional_elements_start and additional_elements add elements randomly.

The last two strategies for adding elements, namely ENUM and RANDOM, allow additional parameters. For ENUM, this is the number of elements added, as well as the method used to generate them. For RANDOM, this is the number of randomly chosen elements. Further, these elements can be required to have a certain minimal and maximal length. If RANDOM is considered as a generating method, it can be subsumed by ENUM.

4. *Orderings*

As will be seen later, orderings play an important rôle in the enumeration process. The strategies presented here will behave differently when they are combined with different orderings. Three different orderings are defined; these are all well founded, total, and admissible. They have been implemented in MRC 1.2, and were used for the coset enumeration.

Definition 3 (Length-lexicographical ordering). Let Σ be a finite alphabet. Let $>$ be a total precedence on Σ. Let $v \equiv v_1 \dots v_n \in \Sigma^*$, for $v_i \in \Sigma$, and let $w \equiv w_1 \dots w_m \in \Sigma^*$, for $w_j \in \Sigma$. The *length-lexicographical ordering* $\succ_{\text{ll}}$ is defined as:

$$\begin{aligned} v \succ_{\text{ll}} w \quad \text{iff} \quad & |v| > |w| \quad \text{or} \\ & (|v| = |w| \text{ and } \exists\, 1 \leqslant i \leqslant n : (v_i > w_i \wedge \forall\, 1 \leqslant j < i : v_j \equiv w_j)). \end{aligned}$$

Definition 4 (Knuth–Bendix ordering). Let Σ be a finite alphabet. Let $>$ be a total precedence on Σ. Let $v \equiv v_1 \dots v_n \in \Sigma^*$, for $v_i \in \Sigma$, and let $w \equiv w_1 \dots w_m \in \Sigma^*$,for $w_j \in \Sigma$. Let $l = \min\{n, m\}$. Let $g : \Sigma \to \mathbb{N}^+$ be a total function attaching a weight to each letter of the alphabet. Then g is defined on words $u \equiv u_1 \dots u_o \in \Sigma^*$, for $u_k \in \Sigma$, as:

$$g : \Sigma^* \to \mathbb{N}_0, \qquad u \mapsto \sum_{k=1}^{o} g(u_k).$$

The *Knuth–Bendix ordering (kbo)* $\succ_{\text{kbo}}$ is defined as:

$$\begin{aligned} v \succ_{\text{kbo}} w \quad \text{iff} \quad & g(v) > g(w) \quad \text{or} \\ & (g(v) = g(w) \text{ and } \exists\, 1 \leqslant i \leqslant l : (v_i > w_i \wedge \forall\, 1 < j \leqslant i : v_j \equiv w_j)). \end{aligned}$$

Definition 5 (Syllable ordering). Let Σ be a finite alphabet. Let $>$ be a total precedence on Σ. Let $\tau : \Sigma \to \{l, r\}$ be a total function, named the *status function*. Let $\max(u)$, where $u \in \Sigma^*$, be the largest letter of the word u with respect to the precedence $>$ on Σ.

For $w \in \Sigma^*$ and $a \in \Sigma$, let $|w|_a$ denote the number of occurences of a in w. The *syllable ordering* $\succ_{\text{syl},\tau}$ is defined as:

$$\begin{aligned} v \succ_{\text{syl},\tau} w \text{ iff } & |v|_{\max(vw)} > |w|_{\max(vw)} \text{ or} \\ & [\ \max(vw) = a \text{ and} \\ & \ |v|_a = |w|_a = n \text{ and} \\ & \ v \equiv v_1 a v_2 a \dots v_n a v_{n+1} \text{ and} \\ & \ w \equiv w_1 a w_2 a \dots w_n a w_{n+1} \text{ and} \\ & \ [[\tau(a) = r \text{ and} \\ & \quad \exists\, i, 1 \leqslant i \leqslant n+1 : (v_i \succ_{\text{syl},\tau} w_i \wedge \forall\, j, i+1 \leqslant j \leqslant n+1 : v_j \equiv w_j)] \\ & \quad \text{or} \\ & \ \ [\tau(a) = l \text{ and} \\ & \quad \exists\, i, 1 \leqslant i \leqslant n+1 : (v_i \succ_{\text{syl},\tau} w_i \wedge \forall\, j, 1 \leqslant j \leqslant i-1 : v_j \equiv w_j)]]]. \end{aligned}$$

Notice that the syllable ordering has a recursive definition. In MRC 1.2, the orderings are implemented according to the definitions given, with the restriction that for the syllable ordering: $\forall\, a \in \Sigma : \tau(a) = l$ or $\forall\, a \in \Sigma : \tau(a) = r$. That is, for all letters of the alphabet, syllables are compared either from left to right, or from right to left.

Group	Presentation
$G^{3,7,-9}$	$a^3 = b^7 = c^{-9} = (ab)^2 = (bc)^2 = (ca)^2 = (abc)^2 = 1$
$(7, 7 \mid 2, 3)$	$a^7 = b^7 = (ab)^2 = (Ab)^3 = 1$
Cox	$a^6 = b^6 = (ab)^2 = (a^2b^2)^2 = (a^3b^3)^5 = 1$
$(30, 30 \mid 3, 10) + a^3b^3$	$a^{30} = b^{30} = (ab)^3 = (Ab)^{10} = a^3b^3 = 1$
$\mathrm{PSL}_3(4)$	$a^5 = b^3 = (ab)^4 = (AbABab)^3$ $= ba^2bA^2BA^2Ba^2baBA = 1$
$B_{2,4}$	$a^4 = b^4 = (ab)^4 = (Ab)^4 = (a^2b)^4 = (ab^2)^4$ $= (a^2b^2)^4 = (Abab)^4 = (aBab)^4 = 1$
S_7	$a^7 = b^2 = (ab)^6 = [a, b]^3 = [a^2, b]^2 = [a^3, b]^2 = 1$
$(4, 6 \mid 2, 12) + [A, b]^3$	$a^4 = b^6 = (ab)^2 = (Ab)^{12} = [A, b]^3 = 1$
$(2, 3, 11; 4)$	$a^2 = b^3 = (ab)^{11} = [a, b]^4 = 1$
J_3	$a^2 = c^2 = b^{15} = (ac)^3 = (bc)^2 = abaB^4ab^3$ $= s^2 = t^2 = (sa)^2 = (sc)^2 = (at)^2 = (bt)^3$ $= b^5tB^5t = sbsB^4 = (ct)^4s = (b^2st)^3 = (B^2ctb^4ct)^2$ $= b^2tBabtB^2a = B^2aB^3ctab^2ctb^3ab^3ctactb^7ab^4ct = 1$
J_3^*	$a^2 = c^2 = b^{15} = (ac)^3 = (bc)^2 = abaB^4ab^3$ $= s^2 = t^2 = (sa)^2 = (sc)^2 = (at)^2 = (bt)^3 = b^5tB^5t$ $= sbsB^4 = (ct)^4s = (b^2st)^3 = (B^2ctb^4ct)^2$ $= b^2tBabtB^2a = B^2aB^3ctab^2ctb^3ab^3ctactb^7ab^4ctZ$ $= z^3 = [a, z] = [b, z] = [c, z] = [s, z] = [t, z] = 1$
Weyl B_6	$a^2 = b^2 = c^2 = d^2 = e^2 = f^2 = (ab)^3 = (ac)^2 = (ad)^2$ $= (ae)^2 = (af)^2 = (bc)^3 = (bd)^2 = (be)^2 = (bf)^2$ $= (cd)^3 = (ce)^2 = (cf)^2 = (de)^3 = (df)^2 = (ef)^4 = 1$

Table 1: Presentations of the groups used for the non-pathological examples

Example	Index	Precedence on Σ
$G^{3,7,-9} \mid E$	504	$C > c > B > b > A > a$
$(7,7 \mid 2,3) \mid E$	1092	$B > b > A > a$
Cox $\mid E$	3000	$B > b > A > a$
$(30,30 \mid 3,10) + a^3b^3 \mid E$	3000	$B > b > A > a$
$\mathrm{PSL}_3(4) \mid \langle a \rangle$	4032	$B > b > A > a$
$B_{2,4} \mid E$	4096	$B > b > A > a$
$S_7 \mid E$	5040	$B > b > A > a$
$(4,6 \mid 2,12) + [A,b]^3 \mid E$	5184	$B > b > A > a$
$(2,3,11;4) \mid E$	6072	$B > b > A > a$
$J_3 \mid \langle a,b,c,s \rangle$	6156	$a > A > b > B > c > C > s > S > t > T$
$J_3^* \mid \langle a,b,c,s \rangle$	18468	$Z > z > T > t > S > s > C > c$
		$> B > b > A > a$
Weyl $B_6 \mid E$	46080	$F > f > E > e > D > d > C > c$
		$> B > b > A > a$

Table 2: Indices and precedences for the non-pathological examples computed

Comparison of terms is frequently needed; for example, for the determination of the head term after each reduction step. Therefore, the time needed to compare two terms with respect to the chosen ordering directly influences the time needed to compute the prefix Gröbner basis. As can be seen from the definitions, the length-lexicographical ordering has at least to compare the lengths of the two terms, and at most to compare the lengths and then each character of the terms. The Knuth–Bendix ordering has at least to compute the sum of each of the two terms, and at most to compute the sums and to compare each character of the term. The syllable ordering has at least to count the number of syllables of each term, and at most in addition to compare the syllables recursively. Thus we see that the expense incurred by the comparison of two terms grows, starting from the length-lexicographical ordering, through the Knuth–Bendix, to the syllable ordering.

5. *Non-pathological examples*

In [5] two types of examples were presented, and the results for two different methods for the Todd–Coxeter enumeration process of cosets (namely Felsch- and HLT-style coset-enumeration procedures) are tabulated. The authors distinguished between pathological and non-pathological cases. An enumeration was called *pathological* if τ_F is significantly greater than one, where $\tau_F = M_F/I$, and where I is the index of the subgroup in the group, and M_F is the maximal number of cosets defined at any instant during the enumeration. The index F indicates Felsch-style enumeration. The pathological examples are studied extensively in the rest of this paper.

In this section we consider the non-pathological examples. The examples are based on the groups that are given, together with their presentation, in Table 1 (see page 87). Inverse elements are given by capital letters.

We computed these examples using the 'level' framework , the strategy NONE, and a length-lexicographical ordering. The examples are given in Table 2 (see page 88), together with the indices and the precedence on the alphabet used for the length-lexicographical ordering.

The results are given in Table 3 (see page 90) for the maximal numbers of cosets defined, and in Table 4 (see page 90) for the total numbers of cosets. These will be summarized next. For both Felsch-style enumeration and our method, the maximal number of cosets defined never exceeds the index. For HLT, this holds for almost all the examples, except for $(4, 6 \mid 2, 12) + [A, b]^3 \mid E$, where 198 more cosets are defined maximally, as well as some examples where this number is one higher than the index. For all the examples, our method defines fewer cosets in total than either Felsch or HLT, HLT being twice as bad for Cox $\mid E$, $\mathrm{PSL}_3(4) \mid \langle a \rangle$, $(4, 6 \mid 2, 12) + [A, b]^3 \mid E$, $J_3 \mid \langle a, b, c, s \rangle$ and $J_3^* \mid \langle a, b, c, s \rangle$. Nevertheless, Felsch-style enumeration is to be preferred in this case, as our method will certainly always be much slower.

6. *Pathological examples and notations*

The pathological examples analyzed and tabulated in [**5**] will now be examined. In [**5**], they were analyzed with respect to Felsch- and HLT-style methods. The Felsch-style method was then developed further, and new results for the improved version were presented in [**8**], together with the old findings. We use these examples, and compare the results achieved there to the results that we obtained by using some of the frameworks and strategies presented in the previous sections. The examples are based on the groups that are given—together with their presentation—in Table 5 (see page 91).

The examples themselves, together with the indices, are given in Table 7 (see page 102). Inverse elements are given by capital letters. The results will be summarized in the following sections, and are tabulated in Appendix A on page 102 ff.

We shall write 'n/m', where n is the maximal and m the total number of cosets defined (see also Section 2.2). For the Knuth–Bendix orderings, we have adopted the following notation: $(a\ 3) > (b\ 1)$ means that a has weight 3, b has weight 1 and $a > b$ if compared lexicographically.

7. *Evaluation of the frameworks and the strategies*

In [**5, 8**], two different methods for the Todd–Coxeter enumeration of cosets (namely the Felsch- and HLT-style coset-enumeration procedures) were studied, and the results for thirteen examples were tabulated. We used these examples to evaluate the 'level' framework, introduced in Section 3.2, and the seven strategies, 1–7, introduced in Section 3.4, for adding elements to the border set.

The evaluation of the 'steps' framework revealed the difficulties associated with this framework. The *recompute* parameter can be chosen adequately only if the result is already known. Further, this parameter does not seem to reflect the evolution of the enumeration.

The same is true for the two remaining strategies, RANDOM and ENUM. It is not easy to choose the number of randomly chosen elements, or to determine the additional requirements that they have to fulfill. The results were not very promising. For ENUM,

Example	Felsch	HLT	NONE
$G^{3,7,-9} \mid E$	504	505	504
$(7,7 \mid 2,3) \mid E$	1092	1093	1092
Cox $\mid E$	3000	3000	3000
$(30,30 \mid 3,10) + a^3b^3 \mid E$	3000	3000	3000
$\mathrm{PSL}_3(4) \mid \langle a \rangle$	4032	4033	4032
$B_{2,4} \mid E$	4096	4097	4096
$S_7 \mid E$	5040	5040	5040
$(4,6 \mid 2,12) + [A,b]^3 \mid E$	5184	5382	5184
$(2,3,11;4) \mid E$	6072	6073	6072
$J_3 \mid \langle a,b,c,s \rangle$	6156	6157	6156
$J_3^* \mid \langle a,b,c,s \rangle$	18468	18468	18468
Weyl $B_6 \mid E$	46080	-	46080

Table 3: Non-pathological cases: maximal number of cosets defined

Example	Felsch	HLT	NONE
$G^{3,7,-9} \mid E$	504	855	504
$(7,7 \mid 2,3) \mid E$	1121	1484	1098
Cox $\mid E$	3000	6654	3000
$(30,30 \mid 3,10) + a^3b^3 \mid E$	3000	5871	3000
$\mathrm{PSL}_3(4) \mid \langle a \rangle$	4655	8134	4048
$B_{2,4} \mid E$	5022	6561	4135
$S_7 \mid E$	5082	6026	5040
$(4,6 \mid 2,12) + [A,b]^3 \mid E$	5187	10892	5184
$(2,3,11;4) \mid E$	6101	9725	6072
$J_3 \mid \langle a,b,c,s \rangle$	6870	13986	6156
$J_3^* \mid \langle a,b,c,s \rangle$	19351	41163	18468
Weyl $B_6 \mid E$	46080	-	46080

Table 4: Non-pathological cases: total number of cosets defined

Group	Presentation
E_1	$TrtRR = RsrSS = StsTT = 1$
(2, 5, 7; 2)	$a^2 = b^5 = (ab)^7 = [a, b]^2 = 1$
$G^{3,7,17}$	$a^3 = b^7 = c^{17} = (ab)^2 = (bc)^2 = (ca)^2 = (abc)^2 = 1$
$\mathrm{PSL}_2(11)$	$a^{11} = b^2 = (ab)^3 = (a^4bA^5b)^2 = 1$
(2, 3, 7; 7)	$a^2 = b^3 = (ab)^7 = [a, b]^7 = 1$
$M_{11}^{(1)}$	$a^{11} = b^5 = c^4 = (a^4c^2)^3 = (bc^2)^2 = (abc)^3 = BabA^4 = CbcB^2 = 1$
(8, 7 | 2, 3)	$a^8 = b^7 = (ab)^2 = (Ab)^3 = 1$
Neu	$a^3 = b^3 = c^3 = (ab)^5 = (Ab)^5 = (ac)^4 = (aC)^4$
	$= aBabCacaC = (bc)^3 = (Bc)^4 = 1$
Cam(3)	$r^2srsR^3S = s^2rsrS^3R = 1$
$G^{3,7,16}$	$a^3 = b^7 = c^{16} = (ab)^2 = (bc)^2 = (ca)^2 = (abc)^2 = 1$
$G(2, 4)$	$BAbaBabA^2 = ABabAbaB^4 = 1$
$G(2, 6)$	$BAbaBabA^2 = ABabAbaB^6 = 1$
$G(3, 3)$	$BAbaBabA^3 = ABabAbaB^3 = 1$

Table 5: Presentations of the groups used for the pathological examples

not only does the number of elements have to be chosen, but also the strategy. The choices that were evaluated were not promising, either. The variations of the level framework gave no better results. Thus, we concentrated on one framework and seven strategies.

A length-lexicographical ordering (see Section 4) was used, with the precedences chosen as depicted in Table 7 (see Appendix A.1, on page 102). The results are presented in Tables 8–11, together with the findings of [**8**]; see Appendix A.2, pages 103–104. The first column shows the best results for Felsch-style enumeration, and the second the results for the Lookahead-HLT style. The third column shows the results for the strategy NONE, while columns 4–6 show either the prefix strategies P-ALL, P-G, and P-R, or the inverse strategies I-ALL, I-R, and I-R-P.

The NONE strategy performs better than HLT for eight out of the thirteen examples in terms of the maximal number of cosets defined, and for ten out of the thirteen examples in terms of the total number of cosets defined. Compared to the best results of the Felsch strategy that were presented in [**8**], it performs better for only two examples in terms of the maximal and total numbers of cosets.

Adding elements makes things worse for most of the examples considered. There are, however, notable exceptions. Adding the inverses of the relators (I-R) reduces the number of cosets to be defined for $G(2,4) \mid E$ and $G(2,6) \mid E$ to about 65%.

Adding all the cyclic permutations of the relators to the set of relators considerably improves the enumeration, as G contains more information. Unfortunately, this also slows down the computation in most cases, as the prefix Gröbner bases to be computed are larger. The results are shown in Tables 12–15 (see Appendix A.3, pages 105–106). In particular, if no elements are added to the border set, we get a performance that is at least as good as (or even much better than) without these permutations. In this case, NONE performs better than HLT, with the exception of $G^{3,7,16} \mid E$, where the maximal number of cosets defined is higher, while there are only half as many cosets defined in total. Compared to the Felsch-style enumeration, eight out of the thirteen examples can be computed by defining fewer cosets. Notable among these are the Macdonald groups $G(n,m) \mid E$, where three to seven times fewer cosets have to be defined.

Now, the addition of elements does not improve the behaviour, with the following exceptions. Adding the inverses of the relators (I-R) reduces the number of cosets to be defined for $E_1 \mid E$ from 157/157 to 97/97, which is about 38% less. The same strategy slightly reduces the numbers for Neu $\mid \langle a, c\rangle$, from 1683/1697 to 1637/1671. The same behaviour is found for $G(2,4) \mid E$, where a reduction from 467/467 to 424/424 is achieved, and for $G(3,3) \mid E$, where the reduction is from 9753/9753 to 9253/9253, whereas for $G(2,6) \mid E$ more cosets have to be enumerated. For $G^{3,7,17} \mid \langle ab, c\rangle$, the I-R-P strategy leads to 945/998 cosets enumerated, compared to 1153/1153 using the strategy NONE. For $G^{3,7,16} \mid E$, all the other strategies perform better than NONE.

8. *The influence of the ordering*

The ordering chosen for the computation of the examples presented in the previous section is a length-lexicographical one. However, this is not necessarily the best one. First of all, it is only one of several different possible length-lexicographical orderings, and was selected more or less randomly. Further, two other types of orderings, namely Knuth–Bendix orderings (kbo), and the two kinds of syllable orderings that were presented in Section 4, have been implemented in Mrc 1.2. This allows us to study the influence of the ordering on the number of cosets to be defined. Here, we ignore the fact that these orderings are more complex, and therefore use more time, than the length-lexicographical one. As has already been mentioned, time and space are not being considered at this stage.

The examples were computed using different kbo, length-lexicographical and syllable orderings. For the length-lexicographical and the syllable orderings, at least one example was computed. The orderings, together with the results, are tabulated in Appendix A.4 on page 107 ff.

We use the following abbreviations: 'kbo-x' means a Knuth–Bendix ordering where x has the greatest weight attached; normally, all the other weights are equal to one. The abbreviation 'll-ZzXxYy' describes the length-lexicographical ordering with precedence $Z > z > X > x > Y > y$, while 'syl-l-ZzXxYy' and 'syl-r-ZzXxYy' describe the syllable orderings with precedence as for the length-lexicographical ordering, the 'l' and 'r' indicating that the syllables are compared from left to right, or right to left, respectively.

8.1. $E_1 \mid E$

The first example reveals two characteristics. We find that for all the orderings computed here, the strategy I-R is better than NONE, followed by I-ALL and P-R. Further, the strategies P-ALL, P-G, P-R, and I-R-P (that is, all the strategies involving prefixes) are equally bad. On the other hand, for all the strategies, the syllable orderings perform better than the Knuth-Bendix and the length-lexicographical orderings. The performance of the latter two strategies is rather similar.

Further, this example shows that the differences between different orderings for the same strategy can be quite large. If we consider the best strategy found using a length-lexicographical ordering, namely I-R, we see that the number of cosets defined ranges from 59/89 to 97/97. Remarkably, the best ordering is syl-r-tsrTSR, while ll-tsrTSR is the one that performs worst. For I-ALL, the values range from 99/99 for syl-l-tsrTSR, to 547/547 for ll-tsrTSR.

8.2. $(2, 5, 7; 2) \mid E$

This example shows that there can be large differences for one strategy, and almost none for others. Using the strategy NONE, all the orderings seem to perform equally well, the number of cosets defined ranging from 143/143 to 163/163 (115%). For P-G, the number of cosets defined ranges from 127/139 for kbo-b, to 276/287 (217%/194%) for ll-BbAa. Remarkably, the strategy P-G, used together with the ordering kbo-b, performs best—while it performs the worst, compared to all other combinations computed, when used together with the ordering ll-BbAa.

Compared to the first example, here we have no one strategy that performs better than the others for all orderings, nor one ordering being better than the others for all strategies. The best orderings with respect to the strategies are kbo-A, kbo-a, and ll-BbAa for NONE, kbo-B for P-ALL, kbo-b for P-G, kbo-b for P-R, ll-BbAa for I-ALL, syl-r-abAB for I-R, and syl-l-abAB for I-R-P. The best strategies with respect to the orderings are NONE for kbo-A, P-ALL for kbo-B, NONE for kbo-a, P-G for kbo-b, I-ALL for ll-BbAa, I-R-P for syl-l-abAB, and I-R for syl-r-abAB.

8.3. $G^{3,7,17} \mid \langle ab, c\rangle$

The third example differs from the first two in that the subgroup is not E but $\langle ab, c\rangle$. At a first glance, the addition of inverse elements does not lead to any difference. But for ll-CcbaBA the number of cosets enumerated differs between I-ALL and NONE, and for both length-lexicographical orderings there are differences between I-R and NONE, and between I-ALL and NONE in the enumeration sequence. Further, P-ALL and I-R-P give similar results for all the orderings except the length-lexicographic ones. This is not surprising, as we get $c = C = \lambda$ and $a = B$ and $b = A$ almost immediately from the subgroup relators. As in the previous example, no one strategy is best for all the orderings, nor is any one ordering best for all the strategies.

8.4. $\mathrm{PSL}_2(11) \mid E$

In this example, the maximal number of cosets defined equals the final number of cosets for most combinations. Thus, only the total number of cosets defined is of interest for most strategies. Strategy NONE is revealed to be the best with respect to all the orderings for this example, the best combinations being those with kbo-A or kbo-a. The total number of cosets defined differs only slightly, ranging from 667 to 684.

For the other strategies, there is a greater variation, and no distinct order to indicate which one performs best.

8.5. $(2, 3, 7; 7) \mid E$

This Coxeter group shows another feature of coset enumeration. No single combination of strategy and ordering is best with respect to both the maximal and total numbers of cosets defined. With respect to the maximal number of cosets defined, the combination P-R and kbo-A (1105/1566) is best, while with respect to the total number of cosets defined we get the best result for the combination P-ALL and kbo-b (1368/1449). There are other combinations that reveal the same behaviour. Overall, no single strategy or ordering is noticeably better or worse than the others.

8.6. $M_{11}^{(1)} \mid \langle a \rangle$

This example has two features: its subgroup is generated by a, and only as many cosets are defined as are seen in the result. As with $\mathrm{PSL}_2(11) \mid E$, the total number of cosets is interesting.

Here, the strategy NONE used with the length-lexicographical ordering is optimal, because exactly 720 cosets are defined in total. Note that this ordering performs worst for I-ALL.

8.7. $(8, 7 \mid 2, 3) \mid \langle a^2, Ab \rangle$

This example gives the best results when using the strategy NONE with kbo-A. The values for NONE and I-ALL are the same for all the orderings, except for the length-lexicographical one. Here, too, different orderings are optimal for different strategies.

8.8. Neu $\mid \langle a, c \rangle$

The same holds for this example; for different strategies, different orderings are optimal.

8.9. $\mathrm{Cam}(3) \mid E$

In this example, we have two winners: strategy NONE and ll-SsRr (with 161/173 cosets defined) and strategy NONE and syl-l-RSrs (with 131/233 cosets), depending on whether the maximal or the total number of cosets defined is considered to be more important. The orderings kbo-R and kbo-r (with 158/211) and kbo-S and kbo-s (with 153/207) lie between those two, and might be preferred. All the other strategies perform slightly (by a factor of 1.5) or considerably (by a factor of 10) worse.

8.10. $G^{3,7,16} \mid E$

Strategy P-R, used together with the ordering kbo-B, enumerates 43703/52149 cosets, and thus performs the best of all combinations computed. Compared with the best Felsch method, it still performs about twice as badly.

8.11. $G(2, 4) \mid E$, $G(2, 6) \mid E$

These two examples are considered in Section 9.2, where the Macdonald Groups $G(2, m)$, with $m = 2, \dots, 11, 15$, are analyzed.

8.12. $G(3,3) \mid E$

Here, all the prefix strategies (P-ALL, P-G, P-R, and I-R-P) perform worse or considerably worse than NONE. Remarkably, I-ALL performs best when used together with the syllable orderings, while being up to seven times worse than NONE for length-lexicographical orderings. Strategy I-R performs as well as, or better than, NONE. Further, it depends on the strategy as to which of the two syllable-left orderings performs better.

8.13. *Summary*

The examples show that a combination of all the parameters has to be considered to find good enumerations. While some examples will always exhibit the best result for one particular strategy, regardless of the ordering chosen, others depend on the right combination of both. In addition, we have to choose whether we want faster execution by defining as few cosets as possible, or to use as little space as possible by obtaining a minimum for the maximal number of cosets defined.

9. *The Macdonald groups*

Having evaluated combinations of several strategies and several orderings for the pathological examples considered, it became clear that both the strategy and the ordering have to be chosen individually for each example. Now, a question arises as to whether there exist *families* of groups whose members behave uniformly with respect to coset enumeration. We explain the concept of such 'families' in Section 9.1. In the Sections 9.2 – 9.4, we examine how the Macdonald groups behave with respect to this concept.

9.1. *Families of groups*

The question raised in the introduction is whether there exist families of groups whose members behave uniformly with respect to coset enumeration. That is, given such a family of groups, can we select a combination of one strategy and one ordering which will perform optimally for all members of this family? While it is fairly unlikely that such a family would exist, there might be families whose members are similar enough for the selected strategy and ordering to be pretty good, even though not necessarily the best, for all the members.

If such families do exist, then one could try to learn how to enumerate cosets efficiently for members of this family, using the following approach. Compute some 1000 combinations for a small member, pick those that perform best, plus some randomly chosen ones, and compute larger examples using these combinations. Of course, this is a good idea only if the computation of such a large set is feasible, and if one of the following two situations is encountered. First, if a large problem is intractable using the resources available, then finding an optimal solution for a smaller example seems to be the best way to find a combination of strategy and ordering that will make it possible to treat the large one. Second, if more than one problem is to be solved for a particular family, then the use of the best combinations can reduce the total time required to find the solutions. Otherwise, it would be more reasonable to compute the examples directly, choosing different combinations of strategies and orderings by hand.

The Macdonald groups $G(n,m)$ (see also [**18**]), which form a syntactic family, were analyzed in order to see if they comprise a family with respect to coset enumeration, too. They are generated by the set of relators $\{BAbaBabA^n, ABabAbaB^m\}$, and all have a finite index, although only a very bad approximation of the index is known:

$$\mathsf{index}(G(n,m)) \mid 27 \cdot (n-1) \cdot (m-1) \cdot (gcd(n,m))^8.$$

However, for $n = 2$ the following holds:

$$\forall m : \mathsf{index}(G(2,m)) = m - 1.$$

9.2. *The Macdonald groups $G(2,m)$*

We performed a coset enumeration of $G(n,m) \mid E, n = 2, m = 2, \ldots, 11, 15$, in order to see whether or not the groups $G(2,m)$ behave uniformly with respect to coset enumeration as implemented in Mrc 1.2. Using the 'level' framework (see Section 3.2), seven strategies (NONE, P-ALL, P-G, P-R, I-ALL, I-R, I-R-P; see Section 3.4) and seven orderings (kbo-A, kbo-B, kbo-a, kbo-b, ll-BbAa, syl-l-BbAa, syl-r-BbAa; see Sections 4 and 8) were combined, and the results were compared. Overall, P-ALL, P-G and I-R-P perform almost equally badly, and especially considerably worse than all the other strategies. This is why they are not considered further for the overview given next. The results are tabulated in Appendix A.5 on page 118 ff. The behaviour with respect to the maximal and the total number of cosets enumerated was slightly different. In our analysis, we have concentrated on the results concerning the total number of cosets enumerated.

For $m = 2$, the results for the orderings do not differ much for any of the strategies selected. The strategy NONE is the best, with 17/17 cosets defined. The next one is I-R, with 42/42 to 47/47 cosets defined. Then comes I-ALL, with 46/46 to 71/71, and finally P-R, with 49/49 to 71/71, I-ALL being better than P-R for all the orderings computed, except for ll-BbAa, where they are equal. Remarkably, no combination of strategies and orderings defines more cosets in total than the maximal number.

For $m = 3$, NONE and I-R perform almost equally well, with fewer than 161 cosets defined, I-R being better for all the orderings except for syllable-right. The two other strategies define between 300 and 650 cosets, the exception being I-ALL together with the syllable-right ordering, which defines 171/192 cosets. This is still worse than for NONE and I-R. I-ALL performs better than P-R for the kbo-A, kbo-a, kbo-b, length-lexicographic and syllable-right ordering.

From this point onwards the performance of I-ALL depends very much on the ordering, with kbo-A, kbo-a and the length-lexicographical ordering being bad, and kbo-B and kbo-b being good, while the behaviour of the syllable orderings varies.

For $m = 4$, the picture is different from that for $m = 3$. While P-R is the worst, and I-R is better than NONE except for the syllable-right ordering, I-ALL is the best for both syllable orderings with respect to the total number of cosets defined, while being close to the other two with respect to the maximal number of cosets defined. For kbo-B and kbo-b, I-ALL performs better than NONE.

For $m = 5$, the picture changes again. Now, NONE and I-R perform almost equally, NONE being better for most orderings this time, and I-R being better than NONE only for kbo-b. For kbo-a, the length-lexicographical ordering and both strategies perform almost equally well. P-R is still the worst, except for the length-lexicographical ordering and the syllable-left ordering, where I-ALL is slightly worse. Remarkably, I-ALL is best for syl-r-BbAa.

For $m = 6$, we get a degree of chaos. For kbo-A, kbo-B and ll-BbAa, the strategy NONE is best, while for the others I-R is best. For the latter, NONE performs considerably worse. I-ALL performs better than I-R for kbo-B, and better than NONE for kbo-b, but otherwise worse than both. This time, P-R performs better than I-ALL for the length-lexicographical and syllable orderings. It is still worse than NONE and I-R for all the orderings except for the syllable-right one, where it is better than NONE.

For $m = 7$, we get a similar picture, but there are the following differences. NONE performs better than I-R for kbo-a. Both perform worse for syllable-left (which performed well for $m = 6$). For the syllable orderings, the strategies NONE, I-ALL and P-R perform equally badly. I-ALL and P-R perform almost equally badly for kbo-A and kbo-a.

For $m = 8$, we have the following differences from $m = 7$. NONE performs better than I-ALL for kbo-b, but still worse than I-R. P-R performs better than I-ALL for kbo-A and kbo-a, but considerably worse for kbo-b.

For $m = 9, 10, 11$, no more changes are observed for the kbo and length-lexicographical orderings, while the syllable orderings are still slowly changing, with all the strategies being rather close together. Remarkably, for m equal to 10 or 11, I-ALL together with syl-l-BbAa is worst, with at least about twice as many cosets defined, compared to all other combinations.

Thus, for $m = 8, \ldots 11, 15$, we get the following picture. For kbo-A, kbo-a and ll-BbAa, we find that NONE is best, followed by I-R, with 50% more cosets defined; then comes P-R, and finally I-ALL, with about 400% more cosets defined than for NONE. The gaps are increasing with increasing m. For kbo-B, we find that NONE is better than I-ALL, for which 50% more cosets are defined, followed by I-R with 75% more cosets defined, and finally P-R with 200% more cosets defined. For kbo-b, strategy I-R is the best, with 25% fewer cosets defined than for NONE, which defines about the same number of cosets for all the kbo orderings. As with kbo-B, strategy I-ALL is next, with about 50% more cosets defined than for NONE, and finally we have P-R, with 450% more cosets defined. While the picture stays the same for all $m = 8, \ldots 11$, we see that for NONE, the number of cosets defined increases by only between 1 and 15 cosets for increasing m. For the other strategies, the number of cosets increases by between 50 and 2000 for increasing m. Finally, there are the two syllable orderings. Here the fact just mentioned comes into play. While for syl-l-BbA, strategy I-R is best for $m = 8, \ldots 11$, it gets nearer to NONE with increasing m. For $m = 15$, NONE is best, followed by I-R, P-R, and I-ALL, while we find that the order is I-R, I-ALL, P-R, and finally NONE, for $m = 8$. A similar result, with a different order, holds for syl-r-BbAa.

9.3. *The Macdonald group* $G(2, 8)$ *with different orderings*

This behaviour is rather discouraging, as good combinations for small m become bad ones for large m, and vice versa. From $m = 8$ on, however, it looks as though the behaviour becomes stable. Thus, choosing $m = 8$ as the starting point for larger examples seems to be a good idea. Therefore $G(2, 8)$ was computed using the strategies NONE, P-R, I-ALL and I-R, combined with 625 kbos and all the length-lexicographical and syllable-left (also 24) orderings. Thus, a total of 2692 combinations were computed and evaluated.

For the length-lexicographical orderings, all the strategies behave pretty uniformly. Strategy NONE enumerates 1343/1343 cosets for all of them, strategy I-R between 1718/1719 and 1780/1780, strategy P-R between 5261/5261 and 5340/5340, and strategy I-ALL between 6725/6725 and 6869/6869 cosets. Notice that the maximal number of cosets defined is equal to the total number of cosets defined for all the combinations except for some combinations with I-R, where one additional coset had to be defined.

For the kbos, there is a greater variation in the results. For the strategy NONE, between 1210/1210 and 1343/1343 cosets were enumerated. The best performance was achieved by the kbo with the following weights: $(a\ 5), (b\ 1), (A\ 5), (B\ 1)$, and the worst by those with equal weights for all the letters (which gives, in fact, a length-lexicographical ordering). That is, all the kbo orderings performed as well as, or better than, the length-lexicographical orderings. The strategy I-R showed a greater variation. Between 1032/1033 and 2357/2359 cosets were enumerated. The best were those kbos that attached a large weight to b, and small weights to the other letters of the alphabet. The best combination of weights was $(a\ 1), (b\ 5), (A\ 1), (B\ 1)$, in contrast to the results for the strategy NONE. The worst combinations have large weights attached to A and B, and small weights to a and b. Strategy I-ALL yielded the following results: between 2067/2067 and 7235/7235 cosets were enumerated. This strategy was better than I-R for some kbos. It also had the largest variation. Good kbos were those with a large weight attached to B, and small weights to the other letters of the alphabet. The worst combinations were those attaching large weights to a and A, and small weights to b and B. Finally, strategy P-R enumerated between 3776/3776 and 6948/6948 cosets together with kbos, thus showing a smaller variation than I-ALL. Good kbos were those with a large weight attached to B and small weights to the other letters of the alphabet, just as for I-ALL. In contrast, however, the bad kbos were those attaching large weights to a and b, and small weights to A and B, just the opposite to the findings for strategy NONE.

Finally, for the syllable orderings, we got the following results. Contrary to the length-lexicographical orderings, for which the results did not differ very much for the different orderings, the results for the syllable orderings show large variations. For NONE we find that the best ordering is syl-l-AbBa, with 1817/1845 cosets defined, while syl-l-abBA is worst, with 12748/14743 cosets defined. For I-R, the best ordering is syl-l-bBAa, with 1453/1660 cosets defined, while syl-l-AaBb is worst, with 13272/14788 cosets defined. For I-ALL, the best ordering is syl-l-baAB, with 1700/1862 cosets defined, while syl-l-BAab is worst, with 18447/20704 cosets defined, and syl-l-bAaB could not be computed at all. Finally, for P-R the best ordering is syl-l-BbaA, with 2389/2399 cosets defined, while syl-l-AbBa is worst, with 4847/4849. Strategy P-R showed the least variation, while for the other strategies the worst syllable orderings are up to 10 times worse than the best one. Further, no individual syllable ordering is either very good or very bad for all the strategies. It should be noted, too, that the best ordering for I-R is the second best for NONE, and vice versa.

To sum up the results, we find that the Knuth–Bendix orderings are the best for strategy NONE, followed by the length-lexicographical orderings. All the syllable orderings are considerably worse for this strategy. Strategy I-R shows similar behaviour, but this time the length-lexicographical orderings enumerate on average as many cosets as the Knuth–Bendix orderings do; that is, about half of them enumerate fewer cosets, while the other half enumerate more cosets. Most of the syllable orderings enumerate more cosets than the Knuth–Bendix orderings. For the strategy I-ALL, we see that almost all the Knuth–Bendix orderings perform better than the length-lexicographical orderings. Nevertheless, the best and worst orderings are syllable orderings. Finally, for strategy P-R, we have findings similar to those for I-R, that the length-lexicographical orderings enumerate on average as many cosets as the Knuth–Bendix orderings. All the syllable orderings are better than the length-lexicographical orderings, and most of them are better than the Knuth–Bendix orderings.

9.4. *The Macdonald groups $G(3, m)$*

Unfortunately, the groups obtained for $n = 3$ already seem to be much harder. While the results for $G(3, 3)$ are still very good compared to Felsch- or HLT-style methods, it has not yet been possible to compute $G(3, 4)$. Here, a tuned version has to be used, which uses less space. However, the results for $n = 3$, $m = 3$ show some similarities to the results obtained for $n = 2$. The strategies NONE and I-R perform best for most of the orderings used, followed by I-ALL (which is even the best for the syllable ordering syl-l-BbAa). However, I-ALL is considerably worse for the length-lexicographical orderings. All the other strategies perform even more badly, producing the best results for the syllable orderings. The best combination was I-ALL/syl-l-BAab, which was second worst for $G(2, 8)$! Here, further investigations will be necessary.

It should nevertheless be noted that $G(3, 3)$, $G(3, -1)$, and $G(-1, -1)$ are all isomorphic to the generalized quaternion group of order 16, and that the coset enumeration for the latter two is almost trivial!

10. *Coset enumeration over general groups: examples*

In Section 2.3 we explained the idea of splitting the set of relators R into two sets of relators $R = R_1 \cup R_2$, such that we can study the index of the subgroup generated by $S \cup N(R_2)$ in $\mathcal{G}_1$, where $\mathcal{G}_1$ is the group presented by $\langle \Sigma, R_1 \rangle$ or $\langle \Sigma, \widetilde{R_1} \rangle$, respectively, instead of studying the index of the subgroup generated by $S \cup N(R)$ in $\mathcal{F}$. This is done in this section by choosing suitable relators from R; that is, by splitting R into $R_1 \cup R_2$, where we can complete R_1 with respect to the ordering chosen, using the Knuth–Bendix completion procedure. Then prefix Gröbner basis techniques in more general group rings can be applied. For example, provided that the relators are of the form σ^n, where $\sigma \in \Sigma$ and $n \in \mathbb{N}^+$, we can use the Knuth–Bendix completion procedure to get a finite, convergent presentation of the underlying group. For the examples $(2, 5, 7; 2) \mid E$, $G^{3,7,17} \mid \langle ab, c \rangle$, $\mathrm{PSL}_2(11) \mid E$, $(2, 3, 7; 7) \mid E$, $M_{11}^{(1)} \mid \langle a \rangle$, $(2, 8 \mid 2, 3) \mid \langle a^2, Ab \rangle$, Neu $\mid \langle a, c \rangle$ and $G^{3,7,16} \mid E$, all the relators having this form were used to define the new groups $\mathcal{G}_1$ in each case (see Table 6 on page 100). The results for the modified coset enumeration using completed group presentations for the respective $\langle \Sigma, R_1 \rangle$ are tabulated in Appendix A.6 on page 124 ff. Notice that each ordering may yield a different complete set of relators $\widetilde{R_1}$ for a given set of relators R_1. As with all other examples computed, no uniform behaviour is detectable. The combination of ordering, framework, and strategy for selecting additional elements influences whether the results are better or worse when compared to the coset enumeration modulo the free group.

For $(2, 5, 7; 2) \mid E$, we generally get better results than before; that is, fewer cosets are enumerated. The best result is now 45/45 cosets, enumerated for syl-l-abAB and I-ALL, compared to 143/143 for kbo-A and NONE.

For $G^{3,7,17} \mid \langle ab, c \rangle$, on the other hand, we get worse results; that is, for most combinations, more cosets have to be enumerated. The best result is now 913/1045 cosets, enumerated for syl-l-CcBbAa and P-R, compared to 821/867, for kbo-a and P-ALL.

For $\mathrm{PSL}_2(11) \mid E$, we get optimal results for all orderings when combined with strategy NONE. All the other combinations are equally good or better, with two exceptions—namely strategy I-R combined with the two syllable orderings.

For $(2, 3, 7; 7) \mid E$, we get better results for most combinations, but there are quite a few combinations that behave worse. The best result improved to 1092/1103 cosets, enumerated

Group	R_1	R_2
$(2, 5, 7; 2)$	$a^2 = b^5 = 1$	$(ab)^7 = [a, b]^2 = 1$
$G^{3,7,17}$	$a^3 = b^7 = c^{17} = 1$	$(ab)^2 = (bc)^2 = (ca)^2 = (abc)^2 = 1$
$\mathrm{PSL}_2(11)$	$a^{11} = b^2 = 1$	$(ab)^3 = (a^4bA^5b)^2 = 1$
$(2, 3, 7; 7)$	$a^2 = b^3 = 1$	$(ab)^7 = [a, b]^7 = 1$
$M_{11}^{(1)}$	$a^{11} = b^5 = c^4 = 1$	$(a^4c^2)^3 = (bc^2)^2 = (abc)^3$ $= BabA^4 = CbcB^2 = 1$
$(8, 7 \mid 2, 3)$	$a^8 = b^7 = 1$	$(ab)^2 = (Ab)^3 = 1$
Neu	$a^3 = b^3 = c^3 = 1$	$(ab)^5 = (Ab)^5 = (ac)^4 = (aC)^4$ $= aBabCacaC = (bc)^3 = (Bc)^4 = 1$
$G^{3,7,16}$	$a^3 = b^7 = c^{16} = 1$	$(ab)^2 = (bc)^2 = (ca)^2 = (abc)^2 = 1$

Table 6: Relators R_1 and R_2 for the general group examples

for kbo-B and NONE, compared to 1105/1566 for kbo-A and P-R.

For $M_{11}^{(1)} \mid \langle a \rangle$, most of the combinations are worse, and only a few are better. The best result, which was already optimal, remained the same.

For $(2, 8 \mid 2, 3) \mid \langle a^2, Ab \rangle$, we have almost as many combinations that are better as there are those that are worse. The best result improved to 448/549 cosets, enumerated for syl-l-abAB and I-R, compared to 766/773 for kbo-A and NONE.

For Neu $\mid \langle a, c \rangle$, the same result holds. The best result improved to 560/656 cosets, enumerated for syl-l-CcBbAa and NONE, compared to 1637/1671 for ll-CcBbAa and I-R.

For $G^{3,7,16} \mid E$, we again have a mix of better and worse combinations. The best result improved to 30949/42538 cosets, enumerated for ll-abcABC and P-ALL, compared to 43931/56621 for syl-l-CcBbAa and P-ALL. But this is still worse than the best results for Felsch. Nevertheless, the results are about as good as the default strategy for Felsch-style enumeration, as described in [8].

Overall, we have the same problem as before. We have no clues as to which combination to choose to get good results for the coset-enumeration process. Nevertheless, we have found an additional parameter to further improve this process. It should be noted that we also have the option of not adding all of the relators having the form σ^n. That is, we can move only some of them (or even other relators), provided that there exists a finite and convergent presentation of the underlying group.

For example, Appendix A.6.3, on page 126, tabulates the results for the example $G^{3,7,17} \mid \langle ab, c \rangle$, but this time we split the set of relators R into $R_1 : a^3 = b^7 = 1$

and $R_2 : c^{17} = (ab)^2 = (bc)^2 = (ca)^2 = (abc)^2 = 1$. This was inspired by the fact that c is one of the generators of the subgroup. As before, some combinations yielded better results, and some worse. The best result was 911/995 cosets, enumerated for kbo-C and P-ALL, and lies between the best result for the standard enumeration (that is, 821/867 for kbo-a and P-ALL) and the best result for the first modified enumeration (913/1045 for syl-l-CcBbAa and P-R).

11. *Summary and possible enhancements*

We have experimented with a new procedure to enumerate cosets; this has led to the use of two frameworks and nine strategies. Additionally, one framework allows a number of variations, which have not as yet been examined more closely. The evaluation was performed using examples from [5], which were used there to assess the performance of Felsch- and HLT-style enumeration procedures. The results obtained for the new method were compared to the results presented in [5] and in [8].

In [5], the examples were divided into two classes: pathological and non-pathological ones. For the non-pathological examples, the new procedure (the ‘level’ framework, the strategy NONE, and a length-lexicographical ordering) performed as well, or better, compared to Felsch- and HLT-style enumeration. For the pathological examples, one framework and seven strategies were further evaluated. For these examples, the new procedure sometimes performed much better, and sometimes worse, than the existing methods. While it was in general better than HLT-style enumeration, there were some examples where it performed worse than Felsch-style enumeration. For the MacDonald groups, our method is currently the best method known for the enumeration of cosets, in terms of numbers of cosets defined, both maximally and in total. For example, two times fewer cosets are enumerated for $G(3,3) \mid E$, and even four times fewer for $G(2,6) \mid E$, compared to the next best method.

In general, the examples considered do not behave uniformly with respect to the new procedure. It depends on the example as to whether there are best strategies or orderings, and whether or not there is a wide variation of the number of cosets enumerated for the combinations evaluated. For most of the examples, the behaviour of the enumeration process is quite variable, depending largely on the combination of strategy and ordering.

Further, the idea of ‘families’ of groups with respect to coset enumeration was evaluated. The example considered showed very varying behaviour with respect to coset enumeration using our procedures for the first members of the family, although the behaviour seems to become stable for larger members.

Finally, the idea of enumeration with respect to general groups instead of the free group was examined. Again, the results varied from being better to being worse, without giving any hints as to the causes.

Several enhancements are now possible. First of all, the number of examples has to be expanded, to give more information about the behaviour of this procedure compared to those that have already been known for quite a while, like HLT- and Felsch-style methods. To do that, the implementation has been specialized in order to be able to compute larger examples, as the time and space requirements are very high for the more-general implementation.

On the other hand, there are still other possible frameworks and strategies, which will have to be evaluated. There are also two more strategies that allow more parameters. Further, an examination of the results of all the methods available for enumerating cosets might lead to further improvements of the known methods, or even to a new method that is superior to the current ones.

The idea of families of groups has to be investigated more closely, too. As the groups $\mathrm{PSL}_2(11)$, $M_{11}^{(1)}$, and Neu are not members of families of defining relations, they cannot be used for testing this concept. The groups E_1 and Cam(3) are such groups, but the next member of these families could not be computed with the resources available, as they are already much harder. From the examples considered in this paper, there remain the Coxeter groups (see [6]), which form syntactic families, and which will have to be analyzed more closely.

Finally, the idea of enumeration over general groups gives scope for further experimentation. The number of examples considered here was quite small. This will have to be expanded, in order to find out whether this idea allows for better enumerations, or even perhaps gives more insight into the coset-enumeration process.

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Appendix A. *Tables of the pathological Todd–Coxeter examples*

Appendix A.1. *Precedences*

The following table lists the index for each of the examples computed together with the precendence on the alphabet used for the length-lexicographical ordering.

Example	Index	Precedence on Σ
$E_1 \mid E$	1	$T > t > S > s > R > r$
$(2,5,7;2) \mid E$	1	$B > b > A > a$
$G^{3,7,17} \mid \langle ab, c\rangle$	1	$C > c > B > b > A > a$
$\mathrm{PSL}_2(11) \mid E$	660	$B > b > A > a$
$(2,3,7;7) \mid E$	1092	$B > b > A > a$
$M_{11}^{(1)} \mid \langle a\rangle$	720	$C > c > B > b > A > a$
$(8,7 \mid 2,3) \mid \langle a^2, Ab\rangle$	448	$B > b > A > a$
Neu $\mid \langle a, c\rangle$	240	$C > c > B > b > A > a$
$\mathrm{Cam}(3) \mid E$	120	$S > s > R > r$
$G^{3,7,16} \mid E$	21504	$C > B > A > c > b > a$
$G(2,4) \mid E$	3	$B > b > A > a$
$G(2,6) \mid E$	5	$B > b > A > a$
$G(3,3) \mid E$	16	$B > b > A > a$

Table 7: Indices and precedences for the pathological examples computed

Appendix A.2. *Without symmetric relators*

Example	Felsch	HLT	NONE	P-ALL	P-G	P-R
$E_1 \mid E$	98	695	584	648	648	1660
$(2,5,7;2) \mid E$	216	224	205	188	187	228
$G^{3,7,17} \mid \langle ab, c\rangle$	724	1381	1153	1857	2528	1321
$\mathrm{PSL}_2(11) \mid E$	660	661	660	715	702	660
$(2,3,7;7) \mid E$	1221	2286	1534	1883	1579	1249
$M_{11}^{(1)} \mid \langle a\rangle$	720	721	720	2070	1057	1130
$(8,7 \mid 2,3) \mid \langle a^2, Ab\rangle$	824	1241	1298	1742	1609	2139
$\mathrm{Neu} \mid \langle a, c\rangle$	2650	4553	4358	15137	8240	6386
$\mathrm{Cam}(3) \mid E$	653	2189	1222	1207	4809	3238
$G^{3,7,16} \mid E$	21504	69990	75162	67023	67080	67918
$G(2,4) \mid E$	3188	2973	3556	1871	8550	5087
$G(2,6) \mid E$	7889	4194	10593	18795	9826	16685
$G(3,3) \mid E$	25481	29007	70627	150972	88335	126650

Table 8: Maximal number of cosets defined: prefix strategies

Example	Felsch	HLT	NONE	P-ALL	P-G	P-R
$E_1 \mid E$	104	758	584	648	648	1680
$(2,5,7;2) \mid E$	216	227	205	196	200	265
$G^{3,7,17} \mid \langle ab, c\rangle$	761	2315	1153	2235	3473	1515
$\mathrm{PSL}_2(11) \mid E$	743	824	684	826	921	879
$(2,3,7;7) \mid E$	1310	2880	1602	1951	1977	1917
$M_{11}^{(1)} \mid \langle a\rangle$	724	1349	801	2448	1524	1612
$(8,7 \mid 2,3) \mid \langle a^2, Ab\rangle$	840	1422	1300	1817	1741	2294
$\mathrm{Neu} \mid \langle a, c\rangle$	2750	7158	4403	15584	8488	6668
$\mathrm{Cam}(3) \mid E$	660	2206	1222	1238	4823	3247
$G^{3,7,16} \mid E$	23702	161805	75705	82283	82351	84529
$G(2,4) \mid E$	3193	3255	3560	1893	8691	5248
$G(2,6) \mid E$	7893	4582	10597	19154	9947	17221
$G(3,3) \mid E$	25496	31993	70859	154763	89951	129594

Table 9: Total number of cosets defined: prefix strategies

Example	Felsch	HLT	NONE	I-ALL	I-R	I-R-P
$E_1 \mid E$	98	695	584	746	730	890
$(2, 5, 7; 2) \mid E$	216	224	205	211	150	193
$G^{3,7,17} \mid \langle ab, c\rangle$	724	1381	1153	1153	1153	1857
$\mathrm{PSL}_2(11) \mid E$	660	661	660	660	660	660
$(2, 3, 7; 7) \mid E$	1221	2286	1534	1743	1534	1898
$M_{11}^{(1)} \mid \langle a\rangle$	720	721	720	1114	720	2174
$(8, 7 \mid 2, 3) \mid \langle a^2, Ab\rangle$	824	1241	1298	1581	1147	1744
$\mathrm{Neu} \mid \langle a, c\rangle$	2650	4553	4358	21975	7027	15184
$\mathrm{Cam}(3) \mid E$	653	2189	1222	1161	2140	1960
$G^{3,7,16} \mid E$	21504	69990	75162	62613	78813	66887
$G(2, 4) \mid E$	3188	2973	3556	12663	2274	2851
$G(2, 6) \mid E$	7889	4194	10593	19571	7147	3647
$G(3, 3) \mid E$	25481	29007	70627	111249	120529	28141

Table 10: Maximal number of cosets defined: inverse strategies

Example	Felsch	HLT	NONE	I-ALL	I-R	I-R-P
$E_1 \mid E$	104	758	584	746	735	893
$(2, 5, 7; 2) \mid E$	216	227	205	215	151	198
$G^{3,7,17} \mid \langle ab, c\rangle$	761	2315	1153	1155	1153	2235
$\mathrm{PSL}_2(11) \mid E$	743	824	684	987	713	742
$(2, 3, 7; 7) \mid E$	1310	2880	1602	1872	1604	1960
$M_{11}^{(1)} \mid \langle a\rangle$	724	1349	801	1479	955	2550
$(8, 7 \mid 2, 3) \mid \langle a^2, Ab\rangle$	840	1422	1300	1742	1162	1822
$\mathrm{Neu} \mid \langle a, c\rangle$	2750	7158	4403	23401	7217	15639
$\mathrm{Cam}(3) \mid E$	660	2206	1222	1168	2171	1971
$G^{3,7,16} \mid E$	23702	161805	75705	90200	81280	82004
$G(2, 4) \mid E$	3193	3255	3560	12835	2283	2880
$G(2, 6) \mid E$	7893	4582	10597	19754	7199	3674
$G(3, 3) \mid E$	25496	31993	70859	112412	121480	28503

Table 11: Total number of cosets defined: inverse strategies

Appendix A.3. *Adding symmetric relators*

Example	Felsch	HLT	NONE	P-ALL	P-G	P-R
$E_1 \mid E$	98	695	157	572	572	572
$(2,5,7;2) \mid E$	216	224	143	276	276	275
$G^{3,7,17} \mid \langle ab, c\rangle$	724	1381	1153	968	1576	2206
$\mathrm{PSL}_2(11) \mid E$	660	661	660	947	888	897
$(2,3,7;7) \mid E$	1221	2286	1534	1561	1518	2613
$M_{11}^{(1)} \mid \langle a\rangle$	720	721	720	720	720	720
$(8,7 \mid 2,3) \mid \langle a^2, Ab\rangle$	824	1241	973	1862	1759	1325
$\mathrm{Neu} \mid \langle a, c\rangle$	2650	4553	1683	3828	2699	3361
$\mathrm{Cam}(3) \mid E$	653	2189	161	1884	1884	1883
$G^{3,7,16} \mid E$	21504	69990	75058	59350	59338	62715
$G(2,4) \mid E$	3188	2973	467	2767	2767	2585
$G(2,6) \mid E$	7889	4194	1343	4746	4746	3904
$G(3,3) \mid E$	25481	29007	9753	76022	76022	69897

Table 12: Maximal number of cosets defined: symmetric prefix strategies

Example	Felsch	HLT	NONE	P-ALL	P-G	P-R
$E_1 \mid E$	104	758	157	572	572	572
$(2,5,7;2) \mid E$	216	227	143	282	287	284
$G^{3,7,17} \mid \langle ab, c\rangle$	761	2315	1153	1026	1887	2492
$\mathrm{PSL}_2(11) \mid E$	743	824	684	1082	1027	1036
$(2,3,7;7) \mid E$	1310	2880	1602	1650	1885	2902
$M_{11}^{(1)} \mid \langle a\rangle$	724	1349	720	1069	860	956
$(8,7 \mid 2,3) \mid \langle a^2, Ab\rangle$	840	1422	975	1936	1834	1389
$\mathrm{Neu} \mid \langle a, c\rangle$	2750	7158	1697	3924	2784	3455
$\mathrm{Cam}(3) \mid E$	660	2206	173	1884	1884	1883
$G^{3,7,16} \mid E$	23702	161805	75453	65472	65460	62716
$G(2,4) \mid E$	3193	3255	467	2767	2767	2585
$G(2,6) \mid E$	7893	4582	1343	4746	4746	3904
$G(3,3) \mid E$	25496	31993	9753	76025	76025	69900

Table 13: Total number of cosets defined: symmetric prefix strategies

Example	Felsch	HLT	NONE	I-ALL	I-R	I-R-P
$E_1 \mid E$	98	695	157	542	97	572
$(2,5,7;2) \mid E$	216	224	143	138	176	277
$G^{3,7,17} \mid \langle ab, c\rangle$	724	1381	1153	1153	1153	945
$\mathrm{PSL}_2(11) \mid E$	660	661	660	660	660	967
$(2,3,7;7) \mid E$	1221	2286	1534	1725	1710	1561
$M_{11}^{(1)} \mid \langle a\rangle$	720	721	720	720	720	720
$(8,7 \mid 2,3) \mid \langle a^2, Ab\rangle$	824	1241	973	1213	1166	1860
$\mathrm{Neu} \mid \langle a, c\rangle$	2650	4553	1683	9589	1637	3869
$\mathrm{Cam}(3) \mid E$	653	2189	161	1922	386	2005
$G^{3,7,16} \mid E$	21504	69990	75058	47841	65109	59115
$G(2,4) \mid E$	3188	2973	467	2544	424	2770
$G(2,6) \mid E$	7889	4194	1343	4517	1481	4731
$G(3,3) \mid E$	25481	29007	9753	57708	9253	76276

Table 14: Maximal number of cosets defined: symmetric inverse strategies

Example	Felsch	HLT	NONE	I-ALL	I-R	I-R-P
$E_1 \mid E$	104	758	157	542	97	572
$(2,5,7;2) \mid E$	216	227	143	161	177	284
$G^{3,7,17} \mid \langle ab, c\rangle$	761	2315	1153	1153	1153	998
$\mathrm{PSL}_2(11) \mid E$	743	824	684	876	698	1113
$(2,3,7;7) \mid E$	1310	2880	1602	1834	1757	1652
$M_{11}^{(1)} \mid \langle a\rangle$	724	1349	720	999	760	1056
$(8,7 \mid 2,3) \mid \langle a^2, Ab\rangle$	840	1422	975	1304	1167	1933
$\mathrm{Neu} \mid \langle a, c\rangle$	2750	7158	1697	10066	1671	3964
$\mathrm{Cam}(3) \mid E$	660	2206	173	1922	391	2005
$G^{3,7,16} \mid E$	23702	161805	75453	71238	66091	65196
$G(2,4) \mid E$	3193	3255	467	2544	424	2770
$G(2,6) \mid E$	7893	4582	1343	4517	1481	4731
$G(3,3) \mid E$	25496	31993	9753	57709	9253	76279

Table 15: Total number of cosets defined: symmetric inverse strategies

Appendix A.4. *Examples: using different orderings*

Appendix A.4.1. $E_1 \mid E$

Ordering	Precedence on Σ
kbo-R	$(r\ 1) > (R\ 6) > (s\ 1) > (S\ 1) > (t\ 1) > (T\ 1)$
kbo-S	$(r\ 1) > (R\ 1) > (s\ 1) > (S\ 6) > (t\ 1) > (T\ 1)$
kbo-T	$(r\ 1) > (R\ 1) > (s\ 1) > (S\ 1) > (t\ 1) > (T\ 6)$
kbo-r	$(r\ 6) > (R\ 1) > (s\ 1) > (S\ 1) > (t\ 1) > (T\ 1)$
kbo-s	$(r\ 1) > (R\ 1) > (s\ 6) > (S\ 1) > (t\ 1) > (T\ 1)$
kbo-t	$(r\ 1) > (R\ 1) > (s\ 1) > (S\ 1) > (t\ 6) > (T\ 1)$
kbo-tsr	$(r\ 6) > (R\ 1) > (s\ 12) > (S\ 1) > (t\ 18) > (T\ 1)$
ll-TtSsRr	$T > t > S > s > R > r$
ll-tsrTSR	$t > s > r > T > S > R$
syl-l-tsrTSR	$t > s > r > T > S > R$
syl-r-tsrTSR	$t > s > r > T > S > R$

Table 16: Orderings for $E_1 \mid E$

Ordering	NONE	P-ALL	P-G	P-R	P-R-2	I-ALL	I-R	I-R-P
kbo-R	146	561	561	538	205	421	96	571
kbo-S	146	504	504	491	205	433	96	545
kbo-T	146	518	518	502	205	427	96	544
kbo-r	146	558	558	553	193	418	89	557
kbo-s	146	542	542	537	193	425	89	533
kbo-t	146	549	549	544	193	431	89	557
kbo-tsr	135	496	496	495	164	312	81	521
ll-TtSsRr	157	572	572	572	223	542	97	572
ll-tsrTSR	157	576	576	576	223	547	97	565
syl-l-tsrTSR	95	360	362	331	86	99	65	370
syl-r-tsrTSR	99	389	388	387	80	114	59	395

Ordering	NONE	P-ALL	P-G	P-R	P-R-2	I-ALL	I-R	I-R-P
kbo-R	146	561	561	538	205	421	96	571
kbo-S	146	504	504	491	205	433	96	545
kbo-T	146	518	518	502	205	427	96	544
kbo-r	146	558	558	553	193	418	89	557
kbo-s	146	542	542	537	193	425	89	533
kbo-t	146	549	549	544	193	431	89	557
kbo-tsr	135	496	496	495	164	312	81	521
ll-TtSsRr	157	572	572	572	223	542	97	572
ll-tsrTSR	157	576	576	576	223	547	97	565
syl-l-tsrTSR	95	360	362	331	86	99	96	370
syl-r-tsrTSR	99	389	388	387	80	114	89	395

Table 17: Maximal/total number of cosets defined: $E_1 \mid E$

Appendix A.4.2. (2, 5, 7; 2) | E

Ordering	Precedence on Σ
kbo-A	$(a\ 1) > (b\ 1) > (A\ 6) > (B\ 1)$
kbo-B	$(a\ 1) > (b\ 1) > (A\ 1) > (B\ 6)$
kbo-a	$(a\ 6) > (b\ 1) > (A\ 1) > (B\ 1)$
kbo-b	$(a\ 1) > (b\ 6) > (A\ 1) > (B\ 1)$
ll-BbAa	$B > b > A > a$
syl-l-abAB	$a > b > A > B$
syl-r-abAB	$a > b > A > B$

Table 18: Orderings for (2, 5, 7; 2) | E

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	143	171	172	171	149	176	171
kbo-B	163	161	162	165	163	177	165
kbo-a	143	162	163	224	151	157	164
kbo-b	163	237	127	133	163	158	237
ll-BbAa	143	276	276	275	138	176	277
syl-l-abAB	148	164	158	260	148	155	147
syl-r-abAB	148	164	158	247	148	147	163

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	143	177	179	178	152	177	178
kbo-B	163	163	165	167	165	178	168
kbo-a	143	170	172	239	155	165	173
kbo-b	163	246	139	141	165	159	248
ll-BbAa	143	282	287	284	161	177	284
syl-l-abAB	151	175	169	279	152	164	156
syl-r-abAB	151	177	171	272	151	159	172

Table 19: Maximal/total number of cosets defined: (2, 5, 7; 2) | E

Appendix A.4.3. $G^{3,7,17} \mid \langle ab, c\rangle$

Ordering	Precedence on Σ
kbo-A	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 6) > (a\ 1)$
kbo-B	$(C\ 1) > (c\ 1) > (B\ 6) > (b\ 1) > (A\ 1) > (a\ 1)$
kbo-C	$(C\ 6) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 1)$
kbo-a	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 6)$
kbo-b	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 6) > (A\ 1) > (a\ 1)$
kbo-c	$(C\ 1) > (c\ 6) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 1)$
ll-CcBbAa	$C > c > B > b > A > a$
ll-CcbaBA	$C > c > b > a > B > A$
syl-l-CcBbAa	$C > c > B > b > A > a$

Table 20: Orderings for $G^{3,7,17} \mid \langle ab, c\rangle$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1153	1640	1875	1198	1153	1153	1640
kbo-B	1125	900	1921	1590	1125	1125	900
kbo-C	1070	1196	1362	2027	1070	1070	1196
kbo-a	1153	821	1778	1001	1153	1153	821
kbo-b	1071	1085	1532	1186	1071	1071	1085
kbo-c	1110	1563	2187	1544	1110	1110	1563
ll-CcBbAa	1153	968	1576	2206	1153	1153	945
ll-CcbaBA	1153	944	1570	2244	1180	1153	945
syl-l-CcBbAa	1004	882	1081	1647	1004	1004	882

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1153	1760	2280	1306	1153	1153	1760
kbo-B	1125	950	2270	1841	1125	1125	950
kbo-C	1075	1293	1489	2193	1075	1075	1293
kbo-a	1153	867	2122	1064	1153	1153	867
kbo-b	1077	1106	1907	1300	1077	1077	1106
kbo-c	1110	1756	2381	1584	1110	1110	1756
ll-CcBbAa	1153	1026	1887	2492	1153	1153	998
ll-CcbaBA	1153	998	1906	2535	1268	1153	999
syl-l-CcBbAa	1040	1012	1198	1853	1040	1040	1012

Table 21: Maximal/total number of cosets defined: $G^{3,7,17} \mid \langle ab, c\rangle$

Appendix A.4.4. $\mathrm{PSL}_2(11) \mid E$

Ordering	Precedence on Σ
kbo-A	$(a\ 1) > (b\ 1) > (A\ 6) > (B\ 1)$
kbo-B	$(a\ 1) > (b\ 1) > (A\ 1) > (B\ 6)$
kbo-a	$(a\ 6) > (b\ 1) > (A\ 1) > (B\ 1)$
kbo-b	$(a\ 1) > (b\ 6) > (A\ 1) > (B\ 1)$
ll-BbAa	$B > b > A > a$
syl-l-abAB	$a > b > A > B$
syl-r-abAB	$a > b > A > B$

Table 22: Orderings for $\mathrm{PSL}_2(11) \mid E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	660	660	660	660	660	660	660
kbo-B	660	1006	1005	1013	660	660	1026
kbo-a	660	661	718	660	660	660	661
kbo-b	660	1027	1030	660	660	660	1008
ll-BbAa	660	947	888	897	660	660	967
syl-l-abAB	660	660	660	666	660	660	660
syl-r-abAB	660	660	660	665	660	660	660

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	667	881	826	875	749	703	921
kbo-B	684	1189	1189	1203	753	731	1221
kbo-a	667	772	1146	808	690	698	772
kbo-b	684	1180	1183	758	1092	734	1153
ll-BbAa	684	1082	1027	1036	876	698	1113
syl-l-abAB	671	851	824	760	686	676	851
syl-r-abAB	671	844	823	747	686	676	844

Table 23: Maximal/total number of cosets defined: $\mathrm{PSL}_2(11) \mid E$

Appendix A.4.5. $(2, 3, 7; 7) \mid E$

Ordering	Precedence on Σ
kbo-A	$(a\ 1) > (b\ 1) > (A\ 6) > (B\ 1)$
kbo-B	$(a\ 1) > (b\ 1) > (A\ 1) > (B\ 6)$
kbo-a	$(a\ 6) > (b\ 1) > (A\ 1) > (B\ 1)$
kbo-b	$(a\ 1) > (b\ 6) > (A\ 1) > (B\ 1)$
ll-BbAa	$B > b > A > a$
syl-l-abAB	$a > b > A > B$
syl-r-abAB	$a > b > A > B$

Table 24: Orderings for $(2, 3, 7; 7) \mid E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1534	2065	2757	1105	1654	1708	2065
kbo-B	1563	1811	1163	1929	1563	1423	1811
kbo-a	1534	2045	2590	1814	1644	1705	2128
kbo-b	1563	1368	2232	2362	1563	1503	1368
ll-BbAa	1534	1561	1518	2613	1725	1710	1561
syl-l-abAB	1485	2027	1514	2549	1485	1394	2032
syl-r-abAB	1476	2034	1641	2456	1476	1750	2035

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1602	2336	3307	1566	1775	1758	2338
kbo-B	1608	2003	1619	2152	1610	1500	2004
kbo-a	1602	2358	3191	2428	1728	1823	2457
kbo-b	1608	1449	3354	3219	1614	1562	1450
ll-BbAa	1602	1650	1885	2902	1834	1757	1652
syl-l-abAB	1607	2443	2168	3422	1612	1571	2459
syl-r-abAB	1576	2402	2253	3251	1583	1882	2414

Table 25: Maximal/total number of cosets defined: $(2, 3, 7; 7) \mid E$

Appendix A.4.6. $M_{11}^{(1)} \mid \langle a \rangle$

Ordering	Precedence on Σ
kbo-A	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 6) > (a\ 1)$
kbo-B	$(C\ 1) > (c\ 1) > (B\ 6) > (b\ 1) > (A\ 1) > (a\ 1)$
kbo-C	$(C\ 6) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 1)$
kbo-a	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 6)$
kbo-b	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 6) > (A\ 1) > (a\ 1)$
kbo-c	$(C\ 1) > (c\ 6) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 1)$
ll-CcBbAa	$C > c > B > b > A > a$
syl-l-CcBbAa	$C > c > B > b > A > a$

Table 26: Orderings for $M_{11}^{(1)} \mid \langle a \rangle$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	720	720	720	720	720	720	720
kbo-B	720	794	720	720	720	720	810
kbo-C	720	720	720	720	720	720	720
kbo-a	720	720	722	720	720	720	720
kbo-b	720	720	720	720	720	720	720
kbo-c	720	720	740	720	720	720	720
ll-CcBbAa	720	720	720	720	720	720	720
syl-l-CcBbAa	720	720	877	720	720	720	720

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	735	913	883	919	736	733	933
kbo-B	726	1212	1003	1009	888	766	1227
kbo-C	725	1035	883	968	730	741	1045
kbo-a	727	955	845	891	728	748	962
kbo-b	729	1064	918	931	976	733	1139
kbo-c	726	1022	957	906	730	740	997
ll-CcBbAa	720	1069	860	956	999	760	1056
syl-l-CcBbAa	772	1112	1304	1060	772	768	1106

Table 27: Maximal/total number of cosets defined: $M_{11}^{(1)} \mid \langle a \rangle$

Appendix A.4.7. $(8, 7 \mid 2, 3) \mid \langle a^2, Ab \rangle$

Ordering	Precedence on Σ
kbo-A	$(a\ 1) > (b\ 1) > (A\ 6) > (B\ 1)$
kbo-B	$(a\ 1) > (b\ 1) > (A\ 1) > (B\ 6)$
kbo-a	$(a\ 6) > (b\ 1) > (A\ 1) > (B\ 1)$
kbo-b	$(a\ 1) > (b\ 6) > (A\ 1) > (B\ 1)$
ll-BbAa	$B > b > A > a$
syl-l-abAB	$a > b > A > B$
syl-r-abAB	$a > b > A > B$

Table 28: Orderings for $(8, 7 \mid 2, 3) \mid \langle a^2, Ab \rangle$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	766	1883	1402	1290	766	1092	1885
kbo-B	1211	1871	1823	1800	1211	1211	1871
kbo-a	901	1839	1634	991	901	901	1839
kbo-b	1211	1804	979	828	1211	1126	1807
ll-BbAa	973	1862	1759	1325	1213	1166	1860
syl-l-abAB	808	1552	907	852	808	837	1552
syl-r-abAB	1096	1566	946	914	1096	946	1566

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	773	1997	1468	1402	773	1094	2000
kbo-B	1213	2006	1956	1932	1213	1213	2006
kbo-a	903	1921	1699	1040	903	903	1921
kbo-b	1213	1890	1022	848	1213	1130	1889
ll-BbAa	975	1936	1834	1389	1304	1167	1933
syl-l-abAB	932	1690	964	945	932	964	1690
syl-r-abAB	1198	1676	965	960	1198	1023	1676

Table 29: Maximal/total number of cosets defined: $(8, 7 \mid 2, 3) \mid \langle a^2, Ab \rangle$

Appendix A.4.8. Neu | $\langle a, c\rangle$

Ordering	Precedence on Σ
kbo-A	$(a\ 1) > (b\ 1) > (c\ 1) > (A\ 6) > (B\ 1) > (C\ 1)$
kbo-B	$(a\ 1) > (b\ 1) > (c\ 1) > (A\ 1) > (B\ 6) > (C\ 1)$
kbo-C	$(a\ 1) > (b\ 1) > (c\ 1) > (A\ 1) > (B\ 1) > (C\ 6)$
kbo-a	$(a\ 6) > (b\ 1) > (c\ 1) > (A\ 1) > (B\ 1) > (C\ 1)$
kbo-b	$(a\ 1) > (b\ 6) > (c\ 1) > (A\ 1) > (B\ 1) > (C\ 1)$
kbo-c	$(a\ 1) > (b\ 1) > (c\ 6) > (A\ 1) > (B\ 1) > (C\ 1)$
ll-CcBbAa	$C > c > B > b > A > a$
syl-l-bBaAcC	$b > B > a > A > c > C$
syl-r-bBaAcC	$b > B > a > A > c > C$

Table 30: Orderings for Neu | $\langle a, c\rangle$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	2356	3607	1701	3524	4404	1873	3596
kbo-B	2053	3491	1955	2700	2053	2308	3537
kbo-C	2134	3492	2326	2611	2134	2910	3498
kbo-a	2473	3354	17874	1917	2061	2682	3353
kbo-b	4107	3491	1640	2559	4107	2650	3488
kbo-c	2254	2740	1775	1917	2254	2455	2808
ll-CcBbAa	1683	3828	2699	3361	9589	1637	3869
syl-l-bBaAcC	1899	2390	1726	1835	1899	1899	2387
syl-r-bBaAcC	2323	6856	5897	4923	2323	2321	6907

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	2369	3717	1795	3633	4473	1898	3705
kbo-B	2075	3607	2061	2819	2077	2413	3667
kbo-C	2151	3627	2441	2720	2157	3068	3631
kbo-a	2488	3458	18426	1988	2080	2744	3457
kbo-b	4119	3616	1738	2652	4119	2680	3612
kbo-c	2264	2815	1860	1994	2268	2517	2889
ll-CcBbAa	1697	3924	2784	3455	10066	1671	3964
syl-l-bBaAcC	2158	2553	1945	2033	2158	2158	2550
syl-r-bBaAcC	2614	7148	6357	5300	2614	2623	7214

Table 31: Maximal/total number of cosets defined: Neu | $\langle a, c\rangle$

Appendix A.4.9. Cam(3) | E

Ordering	Precedence on Σ
kbo-R	$(r\ 1) > (R\ 4) > (s\ 1) > (S\ 1)$
kbo-S	$(r\ 1) > (R\ 1) > (s\ 1) > (S\ 4)$
kbo-r	$(r\ 4) > (R\ 1) > (s\ 1) > (S\ 1)$
kbo-s	$(r\ 1) > (R\ 1) > (s\ 4) > (S\ 1)$
ll-SsRr	$S > s > R > r$
syl-l-RSrs	$R > S > r > s$
syl-r-RSrs	$R > S > r > s$

Table 32: Orderings for Cam(3) | E

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-R	158	1822	1822	1786	1348	345	1916
kbo-S	153	1798	1797	1777	1176	323	1913
kbo-r	158	1820	1820	1730	1380	389	1911
kbo-s	153	1552	1552	1650	1195	374	1146
ll-SsRr	161	1884	1884	1883	1922	386	2005
syl-l-RSrs	131	1365	1357	1353	601	221	1297
syl-r-RSrs	260	1585	1592	1546	267	204	1582
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-R	211	1824	1824	1788	1351	381	1918
kbo-S	207	1801	1800	1780	1191	360	1916
kbo-r	211	1823	1823	1733	1383	424	1914
kbo-s	207	1554	1554	1654	1205	405	1148
ll-SsRr	173	1884	1884	1883	1922	391	2005
syl-l-RSrs	233	1445	1439	1435	685	320	1377
syl-r-RSrs	357	1670	1677	1632	371	295	1667

Table 33: Maximal/total number of cosets defined: Cam(3) | E

Appendix A.4.10. $G^{3,7,16} \mid E$

Ordering	Precedence on Σ
kbo-A	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 6) > (a\ 1)$
kbo-B	$(C\ 1) > (c\ 1) > (B\ 6) > (b\ 1) > (A\ 1) > (a\ 1)$
kbo-C	$(C\ 6) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 1)$
kbo-a	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 6)$
kbo-b	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 6) > (A\ 1) > (a\ 1)$
kbo-c	$(C\ 1) > (c\ 6) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 1)$
ll-CBAcba	$C > B > A > c > b > a$
ll-CcBbAa	$C > c > B > b > A > a$
ll-abcABC	$a > b > c > A > B > C$
syl-l-CcBbAa	$C > c > B > b > A > a$

Table 34: Orderings for $G^{3,7,16} \mid E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	74987	61706	61714	68019	65423	54308	61692
kbo-B	71112	62995	63048	43703	59117	51741	62962
kbo-C	70642	61362	61883	47130	72039	71241	61372
kbo-a	74940	56210	56217	54166	62231	54269	56217
kbo-b	71394	65925	65932	57258	57437	68873	65835
kbo-c	70470	53561	54117	63307	61494	56301	53768
ll-CBAcba	75061	58660	58661	74906	54323	62430	59052
ll-CcBbAa	75058	59350	59338	62715	47841	65109	59115
ll-abcABC	75061	57748	57746	60196	53444	58543	57541
syl-l-CcBbAa	60190	43931	64239	50731	60190	60190	43931

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	75174	68546	68554	76584	78496	55399	68595
kbo-B	71112	71157	71262	52149	59845	52429	71115
kbo-C	71227	68565	69089	54757	72785	71913	68579
kbo-a	74941	61436	61453	59945	74987	55158	61453
kbo-b	72263	70489	70493	66932	59350	70983	70395
kbo-c	70470	59937	60579	65942	61783	56888	60179
ll-CBAcba	75462	65147	65148	84627	78545	63529	65694
ll-CcBbAa	75453	65472	65460	70462	71238	66091	65196
ll-abcABC	75462	63154	63152	67241	75677	59510	62918
syl-l-CcBbAa	62131	56621	76038	63959	62133	62131	56621

Table 35: Maximal/total number of cosets defined: $G^{3,7,16} \mid E$

Appendix A.4.11. $G(3,3) \mid E$

Ordering	Precedence on Σ
kbo-A	$(a\ 1) > (b\ 1) > (A\ 6) > (B\ 1)$
kbo-B	$(a\ 1) > (b\ 1) > (A\ 1) > (B\ 6)$
kbo-a	$(a\ 6) > (b\ 1) > (A\ 1) > (B\ 1)$
kbo-b	$(a\ 1) > (b\ 6) > (A\ 1) > (B\ 1)$
ll-BbAa	$B > b > A > a$
ll-bBaA	$b > B > a > A$
ll-baBA	$b > a > B > A$
syl-l-BAab	$B > A > a > b$
syl-l-BbAa	$B > b > A > a$

Table 36: Orderings for $G(3,3) \mid E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	7064	64874	64876	54914	17013	7279	66690
kbo-B	7052	68366	68366	55100	19619	7205	66958
kbo-a	17045	70016	70021	67562	16920	13410	69489
kbo-b	16955	68991	68991	66409	19350	5755	68875
ll-BbAa	9753	76022	76022	69897	57708	9253	76276
ll-bBaA	9753	76154	76154	72358	59142	8501	75959
ll-baBA	9753	76039	76039	72036	61084	8468	75830
syl-l-BAab	16698	29187	31186	31567	4917	11576	28930
syl-l-BbAa	9352	37251	35049	28502	7899	8217	33033

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	7095	65034	65036	55056	17068	7345	66853
kbo-B	7084	68550	68550	55263	19668	7276	67142
kbo-a	17276	70193	70198	67745	16976	13513	69666
kbo-b	17203	69171	69171	66588	19409	5777	69050
ll-BbAa	9753	76025	76025	69900	57709	9253	76279
ll-bBaA	9753	76158	76158	72362	59145	8501	75963
ll-baBA	9753	76043	76043	72040	61086	8468	75834
syl-l-BAab	26145	30966	32852	32891	5051	18262	30740
syl-l-BbAa	10984	38747	36549	29470	8108	10450	34366

Table 37: Maximal/total number of cosets defined: $G(3,3) \mid E$

Appendix A.5. *Examples: the Macdonald groups $G(2, m) \mid E$*

Appendix A.5.1. Orderings

Ordering	Precedence on Σ
kbo-A	$(a\ 1) > (b\ 1) > (A\ 6) > (B\ 1)$
kbo-B	$(a\ 1) > (b\ 1) > (A\ 1) > (B\ 6)$
kbo-a	$(a\ 6) > (b\ 1) > (A\ 1) > (B\ 1)$
kbo-b	$(a\ 1) > (b\ 6) > (A\ 1) > (B\ 1)$
ll-BbAa	$B > b > A > a$
syl-l-BbAa	$B > b > A > a$
syl-r-BbAa	$B > b > A > a$

Table 38: Orderings for $G(2, m) \mid E$

Appendix A.5.2. $G(2, 2) \mid E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	17	71	71	71	67	47	71
kbo-B	17	71	71	71	67	47	71
kbo-a	17	71	71	71	67	43	71
kbo-b	17	71	71	71	67	43	71
ll-BbAa	17	71	71	71	71	45	71
syl-l-BbAa	17	47	41	49	46	43	52
syl-r-BbAa	17	66	66	61	53	42	66
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	17	71	71	71	67	47	71
kbo-B	17	71	71	71	67	47	71
kbo-a	17	71	71	71	67	43	71
kbo-b	17	71	71	71	67	43	71
ll-BbAa	17	71	71	71	71	45	71
syl-l-BbAa	17	47	41	49	46	43	52
syl-r-BbAa	17	66	66	61	53	42	66

Table 39: Maximal/total number of cosets defined: $G(2, 2) \mid E$

Appendix A.5.3. $G(2,3) \mid E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	152	422	422	449	431	137	319
kbo-B	140	458	442	426	593	137	491
kbo-a	153	770	770	496	473	126	362
kbo-b	137	428	395	499	440	110	503
ll-BbAa	161	367	367	380	345	137	371
syl-l-BbAa	132	680	656	409	433	81	528
syl-r-BbAa	78	430	673	548	171	116	582
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	152	422	422	449	431	137	319
kbo-B	142	463	447	430	596	141	496
kbo-a	153	770	770	496	473	126	362
kbo-b	140	435	402	506	445	110	510
ll-BbAa	161	367	367	380	345	137	371
syl-l-BbAa	132	717	693	438	433	108	565
syl-r-BbAa	96	464	708	577	192	118	619

Table 40: Maximal/total number of cosets defined: $G(2,3) \mid E$

Appendix A.5.4. $G(2,4) \mid E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1073	2713	2713	2512	1971	412	2707
kbo-B	1018	2533	2532	1951	958	458	2543
kbo-a	1072	2716	2716	2522	1980	386	2732
kbo-b	1017	2556	2556	2555	1007	701	2591
ll-BbAa	467	2767	2767	2585	2544	424	2770
syl-l-BbAa	627	2142	2111	1498	476	465	2099
syl-r-BbAa	469	1659	1652	1191	490	572	1715
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1074	2713	2713	2512	1971	412	2707
kbo-B	1022	2533	2532	1951	958	460	2543
kbo-a	1075	2716	2716	2522	1980	386	2732
kbo-b	1020	2556	2556	2555	1007	706	2591
ll-BbAa	467	2767	2767	2585	2544	424	2770
syl-l-BbAa	684	2146	2115	1500	476	615	2103
syl-r-BbAa	594	1661	1655	1195	490	598	1722

Table 41: Maximal/total number of cosets defined: $G(2,4) \mid E$

Appendix A.5.5. $G(2,5) \mid E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1158	3664	3664	3136	2732	1231	3690
kbo-B	1050	3441	3441	2287	1039	1315	3452
kbo-a	1147	3686	3686	3180	2817	1159	3686
kbo-b	1076	3465	3465	3447	1131	758	3493
ll-BbAa	1331	3690	3690	3193	3281	1347	3684
syl-l-BbAa	699	2898	2857	1762	1597	521	2869
syl-r-BbAa	710	2095	2068	1298	477	767	2199
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1158	3664	3664	3136	2732	1231	3690
kbo-B	1052	3441	3441	2287	1039	1328	3452
kbo-a	1149	3686	3686	3180	2817	1160	3686
kbo-b	1077	3465	3465	3447	1131	758	3493
ll-BbAa	1331	3690	3690	3193	3281	1347	3684
syl-l-BbAa	738	2904	2863	1766	1807	764	2875
syl-r-BbAa	735	2097	2071	1303	477	784	2209

Table 42: Maximal/total number of cosets defined: $G(2,5) \mid E$

Appendix A.5.6. $G(2,6) \mid E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1199	4739	4739	3924	3715	1448	4762
kbo-B	1129	4467	4459	2666	1218	1601	4497
kbo-a	3156	4724	4724	3931	3718	1359	4736
kbo-b	2921	4557	4555	4515	1360	836	4574
ll-BbAa	1343	4746	4746	3904	4517	1481	4731
syl-l-BbAa	1203	3692	3604	2031	2246	609	3690
syl-r-BbAa	1576	2543	2487	1408	1712	867	2760
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1199	4739	4739	3924	3715	1448	4762
kbo-B	1131	4467	4459	2666	1218	1614	4497
kbo-a	3165	4724	4724	3931	3718	1359	4736
kbo-b	2952	4557	4555	4515	1360	836	4574
ll-BbAa	1343	4746	4746	3904	4517	1481	4731
syl-l-BbAa	1445	3700	3612	2037	2389	891	3698
syl-r-BbAa	1639	2545	2490	1414	1841	890	2775

Table 43: Maximal/total number of cosets defined: $G(2,6) \mid E$

Appendix A.5.7. $G(2,7) \mid E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1225	5930	5930	4637	4703	1598	5952
kbo-B	1212	5626	5610	3095	1513	1907	5659
kbo-a	1225	5911	5911	4664	4728	1506	5935
kbo-b	3215	5749	5746	5668	1613	917	5754
ll-BbAa	1343	5871	5871	4510	5422	1648	5901
syl-l-BbAa	2591	4560	4418	2303	2244	1669	4582
syl-r-BbAa	1616	3015	2928	1511	1713	965	3336
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1225	5930	5930	4637	4703	1598	5952
kbo-B	1212	5626	5610	3095	1514	1922	5659
kbo-a	1225	5911	5911	4664	4728	1513	5935
kbo-b	3224	5749	5746	5668	1614	918	5754
ll-BbAa	1343	5871	5871	4510	5422	1649	5901
syl-l-BbAa	2989	4570	4428	2311	2388	2000	4592
syl-r-BbAa	1675	3017	2931	1518	1843	994	3356

Table 44: Maximal/total number of cosets defined: $G(2,7) \mid E$

Appendix A.5.8. $G(2,8) \mid E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1255	7239	7239	5530	5756	1803	7274
kbo-B	1248	6875	6875	3576	1815	2165	6904
kbo-a	1257	7190	7190	5497	5758	1707	7202
kbo-b	1248	7019	7016	6843	1867	963	7039
ll-BbAa	1343	7117	7117	5289	6758	1758	7168
syl-l-BbAa	2623	5502	5286	2589	2244	1894	5556
syl-r-BbAa	1634	3521	3399	1605	1714	1063	3969
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1255	7239	7239	5530	5756	1803	7274
kbo-B	1248	6875	6875	3576	1817	2179	6904
kbo-a	1257	7190	7190	5497	5758	1714	7202
kbo-b	1248	7019	7016	6843	1868	964	7039
ll-BbAa	1343	7117	7117	5289	6758	1759	7168
syl-l-BbAa	3005	5514	5298	2599	2389	2264	5568
syl-r-BbAa	1689	3523	3402	1613	1845	1098	3994

Table 45: Maximal/total number of cosets defined: $G(2,8) \mid E$

Appendix A.5.9. $G(2, 9) \mid E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1266	8645	8645	6350	7047	1955	8662
kbo-B	1264	8199	8199	4151	2117	2389	8239
kbo-a	1266	8662	8662	6323	7020	1836	8685
kbo-b	1267	8394	8388	8174	2209	1027	8412
ll-BbAa	1343	8443	8443	6020	7848	1929	8529
syl-l-BbAa	2624	6518	6218	2889	2244	2119	6612
syl-r-BbAa	1636	4061	3904	1701	1715	1161	4650
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1266	8645	8645	6350	7047	1955	8662
kbo-B	1264	8199	8199	4151	2119	2405	8239
kbo-a	1266	8662	8662	6323	7020	1841	8685
kbo-b	1267	8394	8388	8174	2211	1029	8412
ll-BbAa	1343	8443	8443	6020	7848	1934	8529
syl-l-BbAa	3006	6532	6232	2901	2390	2528	6626
syl-r-BbAa	1690	4063	3907	1710	1847	1202	4680

Table 46: Maximal/total number of cosets defined: $G(2, 9) \mid E$

Appendix A.5.10. $G(2, 10) \mid E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1275	10142	10142	7369	8099	2178	10156
kbo-B	1277	9647	9647	4617	2339	2607	9715
kbo-a	1275	10202	10202	7305	8117	2032	10182
kbo-b	1278	9854	9848	9558	2482	1080	9867
ll-BbAa	1343	9959	9959	6991	9467	2037	10010
syl-l-BbAa	2624	7608	7214	3203	17342	2344	7750
syl-r-BbAa	1636	4635	4439	1799	1716	1259	5379
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1275	10142	10142	7369	8099	2178	10156
kbo-B	1278	9647	9647	4617	2342	2627	9715
kbo-a	1275	10202	10202	7305	8117	2040	10182
kbo-b	1279	9854	9848	9558	2485	1083	9867
ll-BbAa	1343	9959	9959	6991	9467	2042	10010
syl-l-BbAa	3006	7624	7230	3217	18628	2792	7766
syl-r-BbAa	1691	4637	4442	1809	1849	1306	5414

Table 47: Maximal/total number of cosets defined: $G(2, 10) \mid E$

Appendix A.5.11. $G(2, 11) \mid E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1279	11830	11833	8239	9677	2278	11818
kbo-B	1280	11114	11114	4983	2582	2806	11180
kbo-a	1279	11818	11818	8096	9644	2133	11792
kbo-b	1281	11410	11403	10949	2642	1092	11433
ll-BbAa	1343	11498	11498	7820	10689	2202	11604
syl-l-BbAa	2624	8772	8274	3531	17973	2569	8970
syl-r-BbAa	1636	5243	5004	1899	1717	1357	6156
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1279	11830	11833	8239	9677	2278	11818
kbo-B	1282	11114	11114	4983	2586	2830	11180
kbo-a	1279	11818	11818	8096	9644	2142	11792
kbo-b	1283	11410	11403	10949	2645	1096	11433
ll-BbAa	1343	11498	11498	7820	10689	2211	11604
syl-l-BbAa	3007	8790	8292	3547	19260	3056	8988
syl-r-BbAa	1692	5245	5007	1910	1851	1410	6196

Table 48: Maximal/total number of cosets defined: $G(2, 11) \mid E$

Appendix A.5.12. $G(2, 15) \mid E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1285	19459	19459	12476	15979	2875	19489
kbo-B	1326	18410	18410	7141	3604	3682	18540
kbo-a	1285	19336	19336	12197	16000	2675	19387
kbo-b	1326	18931	18925	17614	3644	1263	18955
ll-BbAa	1343	18961	18961	12062	17617	2734	19135
syl-l-BbAa	2624	14168	13154	4983	20178	3469	14670
syl-r-BbAa	1721	1749	9744	1636	8015	7564	2319
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1286	19459	19459	12476	15979	2877	19489
kbo-B	1332	18410	18410	7141	3611	3718	18540
kbo-a	1286	19336	19336	12197	16000	2696	19387
kbo-b	1331	18931	18925	17614	3650	1270	18955
ll-BbAa	1344	18961	18961	12062	17617	2755	19135
syl-l-BbAa	3011	14194	13180	5007	21469	4112	14696
syl-r-BbAa	1859	1826	9804	1696	8017	7567	2334

Table 49: Maximal/total number of cosets defined: $G(2, 15) \mid E$

Appendix A.6. *Examples: coset enumeration over general groups*

Appendix A.6.1. $(2, 5, 7; 2) \mid E$

Ordering	Precedence on Σ
kbo-A	$(a\ 1) > (b\ 1) > (A\ 6) > (B\ 1)$
kbo-B	$(a\ 1) > (b\ 1) > (A\ 1) > (B\ 6)$
kbo-a	$(a\ 6) > (b\ 1) > (A\ 1) > (B\ 1)$
kbo-b	$(a\ 1) > (b\ 6) > (A\ 1) > (B\ 1)$
ll-BbAa	$B > b > A > a$
ll-abAB	$a > b > A > B$
syl-l-abAB	$a > b > A > B$
syl-r-abAB	$a > b > A > B$

Table 50: Orderings for $(2, 5, 7; 2) \mid E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	94	196	196	160	159	136	196
kbo-B	70	132	132	132	104	79	124
kbo-a	94	174	174	152	154	129	178
kbo-b	46	84	84	84	92	78	93
ll-BbAa	94	196	196	160	159	136	196
ll-abAB	94	179	179	154	155	129	179
syl-l-abAB	70	159	153	149	45	83	161
syl-r-abAB	70	137	134	132	114	81	144

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	94	196	196	160	159	136	196
kbo-B	70	132	132	132	104	79	124
kbo-a	94	174	174	152	154	129	178
kbo-b	46	84	84	84	92	78	93
ll-BbAa	94	196	196	160	159	136	196
ll-abAB	94	179	179	154	155	129	179
syl-l-abAB	70	159	153	149	45	83	161
syl-r-abAB	70	145	142	140	116	81	152

Table 51: Maximal/total number of cosets defined: $(2, 5, 7; 2) \mid E$

Appendix A.6.2. $G^{3,7,17} \mid \langle ab, c \rangle$

Ordering	Precedence on Σ
kbo-A	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 6) > (a\ 1)$
kbo-B	$(C\ 1) > (c\ 1) > (B\ 6) > (b\ 1) > (A\ 1) > (a\ 1)$
kbo-C	$(C\ 6) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 1)$
kbo-a	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 6)$
kbo-b	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 6) > (A\ 1) > (a\ 1)$
kbo-c	$(C\ 1) > (c\ 6) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 1)$
ll-CcBbAa	$C > c > B > b > A > a$
ll-CcbaBA	$C > c > b > a > B > A$
syl-l-CcBbAa	$C > c > B > b > A > a$

Table 52: Orderings for $G^{3,7,17} \mid \langle ab, c \rangle$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1153	3936	3618	2308	960	1153	3936
kbo-B	1125	2819	2516	2435	1125	1125	2819
kbo-C	1068	1329	1583	1145	1068	1068	1329
kbo-a	1153	1083	3262	1245	1153	1153	1083
kbo-b	1068	2313	1863	1469	1068	1068	2313
kbo-c	1110	3615	1511	1627	1110	1110	3615
ll-CcBbAa	1153	2493	3187	1297	1154	1153	2493
ll-CcbaBA	1153	2488	1573	1283	1163	1153	2488
syl-l-CcBbAa	1097	3093	1458	913	1097	1097	3093
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1153	4722	4370	2777	1006	1153	4722
kbo-B	1125	3107	3299	3236	1125	1125	3107
kbo-C	1068	1404	1766	1282	1068	1068	1404
kbo-a	1153	1122	4143	1589	1153	1153	1122
kbo-b	1068	2445	2193	1783	1068	1068	2445
kbo-c	1110	3947	1664	1752	1110	1110	3947
ll-CcBbAa	1153	2697	4425	1596	1170	1153	2697
ll-CcbaBA	1153	2699	1908	1584	1200	1153	2699
syl-l-CcBbAa	1124	3569	1788	1045	1124	1124	3569

Table 53: Maximal/total number of cosets defined: $G^{3,7,17} \mid \langle ab, c \rangle$

Appendix A.6.3. $G^{3,7,17} \mid \langle ab, c\rangle$ Variant 2

Ordering	Precedence on Σ
kbo-A	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 6) > (a\ 1)$
kbo-B	$(C\ 1) > (c\ 1) > (B\ 6) > (b\ 1) > (A\ 1) > (a\ 1)$
kbo-C	$(C\ 6) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 1)$
kbo-a	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 6)$
kbo-b	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 6) > (A\ 1) > (a\ 1)$
kbo-c	$(C\ 1) > (c\ 6) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 1)$
ll-CcBbAa	$C > c > B > b > A > a$
ll-CcbaBA	$C > c > b > a > B > A$
syl-l-CcBbAa	$C > c > B > b > A > a$

Table 54: Orderings for $G^{3,7,17} \mid \langle ab, c\rangle$ - B

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1153	1125	1819	1501	960	1153	1125
kbo-B	1125	1287	2310	1185	1125	1125	1287
kbo-C	1068	911	1053	1912	1068	1068	911
kbo-a	1153	1060	1001	1656	1153	1153	1060
kbo-b	1068	1565	1605	1665	1068	1068	1565
kbo-c	1110	920	1492	2000	1110	1110	920
ll-CcBbAa	1153	1143	1601	1105	1154	1153	1143
ll-CcbaBA	1153	1089	1954	1105	1163	1153	1089
syl-l-CcBbAa	1097	1348	1591	2148	1097	1097	1348

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1153	1181	2141	1705	1006	1153	1181
kbo-B	1125	1375	2866	1341	1125	1125	1375
kbo-C	1068	995	1140	2048	1068	1068	995
kbo-a	1153	1087	1131	1853	1153	1153	1087
kbo-b	1068	1592	1920	1835	1068	1068	1592
kbo-c	1110	981	1606	2073	1110	1110	981
ll-CcBbAa	1153	1205	1957	1174	1170	1153	1205
ll-CcbaBA	1153	1135	2390	1174	1200	1153	1135
syl-l-CcBbAa	1124	1588	1904	2433	1124	1124	1588

Table 55: Maximal/total number of cosets defined: $G^{3,7,17} \mid \langle ab, c\rangle$ - B

Appendix A.6.4. $\mathrm{PSL}_2(11) \mid E$

Ordering	Precedence on Σ
kbo-A	$(a\ 1) > (b\ 1) > (A\ 6) > (B\ 1)$
kbo-B	$(a\ 1) > (b\ 1) > (A\ 1) > (B\ 6)$
kbo-a	$(a\ 6) > (b\ 1) > (A\ 1) > (B\ 1)$
kbo-b	$(a\ 1) > (b\ 6) > (A\ 1) > (B\ 1)$
ll-BbAa	$B > b > A > a$
syl-l-abAB	$a > b > A > B$
syl-r-abAB	$a > b > A > B$

Table 56: Orderings for $\mathrm{PSL}_2(11) \mid E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	660	660	660	660	660	660	660
kbo-B	660	660	660	660	660	660	660
kbo-a	660	660	660	660	660	660	660
kbo-b	660	660	660	660	660	660	660
ll-BbAa	660	660	660	660	660	660	660
syl-l-abAB	660	660	660	660	660	660	660
syl-r-abAB	660	660	660	660	660	660	660

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	660	680	681	679	681	678	686
kbo-B	660	681	681	681	676	666	680
kbo-a	660	721	721	721	680	681	722
kbo-b	660	689	689	689	681	666	689
ll-BbAa	660	689	689	689	681	666	689
syl-l-abAB	660	731	731	731	682	682	732
syl-r-abAB	660	721	721	719	681	682	726

Table 57: Maximal/total number of cosets defined: $\mathrm{PSL}_2(11) \mid E$

Appendix A.6.5. $(2, 3, 7; 7) \mid E$

Ordering	Precedence on Σ
kbo-A	$(a\ 1) > (b\ 1) > (A\ 6) > (B\ 1)$
kbo-B	$(a\ 1) > (b\ 1) > (A\ 1) > (B\ 6)$
kbo-a	$(a\ 6) > (b\ 1) > (A\ 1) > (B\ 1)$
kbo-b	$(a\ 1) > (b\ 6) > (A\ 1) > (B\ 1)$
ll-BbAa	$B > b > A > a$
syl-l-abAB	$a > b > A > B$
syl-r-abAB	$a > b > A > B$

Table 58: Orderings for $(2, 3, 7; 7) \mid E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1092	2538	2538	2278	1611	1092	2135
kbo-B	1092	1762	1761	1752	1530	1092	1734
kbo-a	1092	2420	2420	2298	1617	1092	2096
kbo-b	1092	1600	1600	1601	1545	1092	1605
ll-BbAa	1092	2538	2538	2278	1611	1092	2135
syl-l-abAB	1092	1837	1837	1846	1653	1092	1747
syl-r-abAB	1092	1664	1664	1647	1393	1092	1646

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1147	2581	2581	2321	1793	1264	2178
kbo-B	1103	1777	1776	1767	1912	1242	1749
kbo-a	1147	2464	2464	2342	1801	1266	2140
kbo-b	1103	1617	1617	1618	1982	1184	1622
ll-BbAa	1147	2581	2581	2321	1793	1264	2178
syl-l-abAB	1104	1850	1850	1859	1864	1201	1760
syl-r-abAB	1104	1673	1673	1656	1604	1197	1655

Table 59: Maximal/total number of cosets defined: $(2, 3, 7; 7) \mid E$

Appendix A.6.6. $M_{11}^{(1)} \mid \langle a \rangle$

Ordering	Precedence on Σ
kbo-A	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 6) > (a\ 1)$
kbo-B	$(C\ 1) > (c\ 1) > (B\ 6) > (b\ 1) > (A\ 1) > (a\ 1)$
kbo-C	$(C\ 6) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 1)$
kbo-a	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 6)$
kbo-b	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 6) > (A\ 1) > (a\ 1)$
kbo-c	$(C\ 1) > (c\ 6) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 1)$
ll-CcBbAa	$C > c > B > b > A > a$
syl-l-CcBbAa	$C > c > B > b > A > a$

Table 60: Orderings for $M_{11}^{(1)} \mid \langle a \rangle$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	720	720	720	720	720	720	720
kbo-B	720	1037	756	1059	720	720	1064
kbo-C	720	821	720	720	814	720	791
kbo-a	720	720	720	720	720	720	720
kbo-b	720	1131	720	997	720	720	1156
kbo-c	720	1110	720	832	720	720	1309
ll-CcBbAa	720	776	720	720	720	720	744
syl-l-CcBbAa	720	720	721	720	720	720	780

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	720	966	835	925	727	726	973
kbo-B	720	1324	1257	1473	801	737	1348
kbo-C	754	1191	955	1040	1385	741	1152
kbo-a	722	1109	907	1072	723	727	1031
kbo-b	720	1417	1132	1344	761	733	1435
kbo-c	734	1399	1057	1191	830	802	1596
ll-CcBbAa	720	1149	929	1023	826	731	1113
syl-l-CcBbAa	743	1088	830	1036	759	760	1163

Table 61: Maximal/total number of cosets defined: $M_{11}^{(1)} \mid \langle a \rangle$

Appendix A.6.7. $(8, 7 \mid 2, 3) \mid \langle a^2, Ab \rangle$

Ordering	Precedence on Σ
kbo-A	$(a\ 1) > (b\ 1) > (A\ 6) > (B\ 1)$
kbo-B	$(a\ 1) > (b\ 1) > (A\ 1) > (B\ 6)$
kbo-a	$(a\ 6) > (b\ 1) > (A\ 1) > (B\ 1)$
kbo-b	$(a\ 1) > (b\ 6) > (A\ 1) > (B\ 1)$
ll-BbAa	$B > b > A > a$
syl-l-abAB	$a > b > A > B$
syl-r-abAB	$a > b > A > B$

Table 62: Orderings for $(8, 7 \mid 2, 3) \mid \langle a^2, Ab \rangle$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	766	1270	776	1088	775	1092	1270
kbo-B	1211	1431	1436	1153	1271	1211	1431
kbo-a	901	554	1124	1212	929	901	554
kbo-b	657	831	1629	1821	657	716	826
ll-BbAa	973	1409	1378	916	957	1166	1409
syl-l-abAB	566	2272	1475	2251	460	448	1405
syl-r-abAB	607	500	505	497	464	551	495

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	773	1375	897	1224	796	1094	1375
kbo-B	1213	1511	1517	1224	1277	1213	1511
kbo-a	903	690	1207	1289	933	903	690
kbo-b	681	914	1751	1925	681	735	921
ll-BbAa	975	1509	1470	991	1110	1167	1509
syl-l-abAB	676	2354	1506	2333	651	549	1499
syl-r-abAB	710	635	627	631	607	652	629

Table 63: Maximal/total number of cosets defined: $(8, 7 \mid 2, 3) \mid \langle a^2, Ab \rangle$

Appendix A.6.8. Neu | $\langle a, c \rangle$

Ordering	Precedence on Σ
kbo-A	$(a\ 1) > (b\ 1) > (c\ 1) > (A\ 6) > (B\ 1) > (C\ 1)$
kbo-B	$(a\ 1) > (b\ 1) > (c\ 1) > (A\ 1) > (B\ 6) > (C\ 1)$
kbo-C	$(a\ 1) > (b\ 1) > (c\ 1) > (A\ 1) > (B\ 1) > (C\ 6)$
kbo-a	$(a\ 6) > (b\ 1) > (c\ 1) > (A\ 1) > (B\ 1) > (C\ 1)$
kbo-b	$(a\ 1) > (b\ 6) > (c\ 1) > (A\ 1) > (B\ 1) > (C\ 1)$
kbo-c	$(a\ 1) > (b\ 1) > (c\ 6) > (A\ 1) > (B\ 1) > (C\ 1)$
ll-CcBbAa	$C > c > B > b > A > a$
syl-l-bBaAcC	$b > B > a > A > c > C$
syl-r-bBaAcC	$b > B > a > A > c > C$

Table 64: Orderings for Neu | $\langle a, c \rangle$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1487	3950	2421	3930	3032	1221	3927
kbo-B	1446	7602	5956	7000	5392	1845	7230
kbo-C	1364	4466	2800	4076	640	1045	4604
kbo-a	1489	4984	3055	4868	3559	1563	5161
kbo-b	3103	6791	4843	6462	1869	1972	6775
kbo-c	1249	4407	2570	3835	3798	1451	4852
ll-CcBbAa	1677	3412	2448	3181	5357	3327	3386
syl-l-bBaAcC	560	4682	1691	4548	560	567	4690
syl-r-bBaAcC	920	6571	2836	6429	919	920	6551
Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	1488	4006	2495	3986	3154	1239	3983
kbo-B	1446	7632	6022	7029	5610	1871	7258
kbo-C	1365	4507	2863	4122	779	1075	4670
kbo-a	1489	4998	3090	4882	3639	1573	5169
kbo-b	3107	6831	4932	6505	2010	1987	6815
kbo-c	1249	4444	2635	3868	3994	1467	4899
ll-CcBbAa	1677	3442	2503	3221	5487	3332	3416
syl-l-bBaAcC	656	4730	1806	4601	656	661	4738
syl-r-bBaAcC	1000	6627	2901	6486	1008	1003	6604

Table 65: Maximal/total number of cosets defined: Neu | $\langle a, c \rangle$

Appendix A.6.9. $G^{3,7,16} \mid E$

Ordering	Precedence on Σ
kbo-A	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 6) > (a\ 1)$
kbo-B	$(C\ 1) > (c\ 1) > (B\ 6) > (b\ 1) > (A\ 1) > (a\ 1)$
kbo-C	$(C\ 6) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 1)$
kbo-a	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 6)$
kbo-b	$(C\ 1) > (c\ 1) > (B\ 1) > (b\ 6) > (A\ 1) > (a\ 1)$
kbo-c	$(C\ 1) > (c\ 6) > (B\ 1) > (b\ 1) > (A\ 1) > (a\ 1)$
ll-CBAcba	$C > B > A > c > b > a$
ll-CcBbAa	$C > c > B > b > A > a$
ll-abcABC	$a > b > c > A > B > C$
syl-l-CcBbAa	$C > c > B > b > A > a$

Table 66: Orderings for $G^{3,7,16} \mid E$

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	74938	54136	47685	62461	46597	74938	54136
kbo-B	71112	101749	47746	77220	55825	71112	100808
kbo-C	70470	0	0	55223	64020	70470	0
kbo-a	74931	0	47956	53066	36945	74931	0
kbo-b	71112	39664	45387	0	67592	74889	39635
kbo-c	70470	49986	0	53847	57000	73388	50003
ll-CBAcba	74940	51640	47886	0	47116	74940	51640
ll-CcBbAa	74938	49383	45978	0	34493	77056	49006
ll-abcABC	74940	30949	0	65754	0	79455	31944
syl-l-CcBbAa	67074	91754	58808	61699	59542	67074	91754

Ordering	NONE	P-ALL	P-G	P-R	I-ALL	I-R	I-R-P
kbo-A	74939	65117	60207	75726	59618	74939	65117
kbo-B	71112	116942	60063	102463	60694	71112	116096
kbo-C	70470	0	0	63593	66390	70470	0
kbo-a	74931	0	59279	73013	50295	74931	0
kbo-b	71112	47385	57355	0	69063	75283	47315
kbo-c	70470	60362	0	59353	58740	73406	60380
ll-CBAcba	74942	61032	61795	0	67131	74942	61032
ll-CcBbAa	74938	55786	53176	0	56745	77090	55363
ll-abcABC	74942	42539	0	77345	0	79512	43389
syl-l-CcBbAa	68595	113844	74129	73382	61285	68595	113844

Table 67: Maximal/total number of cosets defined: $G^{3,7,16} \mid E$

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