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NOTE ON THE CONTRIBUTION OF LOW ZEROS TO WEIL'S EXPLICIT FORMULA FOR MINIMAL DISCRIMINANTS

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Abstract

The bulk of this paper consists of tables giving lower bounds for discriminants of number fields up to 48. The lower bounds are obtained by using two different inequalities for the discriminant, one due to Odlyzko, and the other due to Serre. These inequalities are derived from Weil's explicit formula by choosing suitable weight functions. The bounds are compared with actual values of the discriminants, and the relative errors are computed. The computations show that, at least for the values computed, the bounds obtained via Odlyzko's inequality are better than those obtained via Serre's inequality, and are generally within a few percentage points of the true value. This difference can be attributed to a difference in the weighting given to the contribution of low zeros by the two inequalities.

1. *Introduction*

Let K be a number field of degree n and signature (r_1, r_2). We denote by d_K the discriminant of K. In 1890, Minkowski proved the inequality [9]

$$|\, d_K| > \frac{\pi}{3}\left(\frac{3\pi}{4}\right)^{n-1}, \tag{1}$$

which shows that the absolute values of discriminants grow to infinity with the degree. Other mathematicians (for example, Mulholland; see [3]) improved this lower bound by using geometric methods. Several years later, using analytic methods and more particularly the Landau identity on the Dedekind zeta function, Stark and Odlyzko obtained better lower bounds. All these identities, as Serre noticed, are only special cases of Weil's explicit formulae relative to the choice of positive functions, F. These formulae relate the discriminant of a number field to two sums: the first sum is positive and runs over all the prime ideals of the field, whereas the other sum runs over all the non-trivial zeros of the Dedekind zeta function attached to the field. In the absence of any knowledge about the location of these zeros, we make this sum positive in order to obtain a lower bound for the discriminant. Since non-trivial zeros of the Dedekind zeta function appear in explicit formulae, it is not surprising that the results obtained are better if we assume the generalised Riemann hypothesis (GRH).

Under (GRH), by considering the function $F_y(x) = e^{-yx^2}$, where $y > 0$ (used by Serre), Poitou (see [7]) established the inequality:

$$\frac{1}{n}\ln(|\, d_K|) \geqslant \ln(8\pi) + \gamma + \frac{r_1}{n}\frac{\pi}{2} - \frac{3}{\ln(n)}, \tag{2}$$

where $\gamma = 0.57721566\ldots$ is the Euler constant.

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If we consider Odlyzko's even function, defined for $x \geqslant 0$ by:

$$F_y(x) = \begin{cases} (1 - x\sqrt{y})\cos(\pi x\sqrt{y}) + \frac{1}{\pi}\sin(\pi x\sqrt{y}), & \text{for } x \in [0, 1/\sqrt{y}] \\ 0, & \text{for } x > 1/\sqrt{y}, \end{cases}$$

we obtain a better estimate (in $1/\ln(n)^2$ instead of $1/\ln(n)$):

$$\frac{1}{n}\ln(|d_K|) \geqslant \ln(8\pi) + \gamma + \frac{r_1}{n}\frac{\pi}{2} - \frac{2\pi^2\left(\lambda(3) + \frac{r_1}{n}\beta(3)\right)}{\ln(n)^2}, \tag{3}$$

where

$$\lambda(3) = 1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \cdots = 1.05179\ldots,$$

and

$$\beta(3) = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32} = 0.96894\ldots.$$

An open problem (see [4]) is to compute, for each degree and signature, the fields $K_{\min}$ having discriminant $d_{K_{\min}}$ of minimal absolute value. For these fields, we can compare the value $\ln|d_{K_{\min}}|$ to Odlyzko's bound B_o for the corresponding degree and signature. For degrees less than or equal to 7, the fields of the minimal discriminants are known. At this time, beyond degree 9, there is a table of the smallest known discriminants of totally imaginary number fields of degree up to 80 (see [1]). We compute the error

$$e = \frac{|d_{K_{\min}}|^{1/n} - \exp(B_o/n)}{\exp(B_o/n)},$$

and we notice that the relative error does not exceed 7% except for the totally real number field of degree 7 of minimal discriminant, for which the error is 14.9%.

It should be noted that this error is due on the one hand to local corrections coming from the decomposition of ideals in the number field, and on the other hand to the contribution of the zeros of the Dedekind zeta function. For the computation of low zeros, there is a formula of Friedman–Lavrik (see [2]) implemented by Tollis (see [10]), but computations are restricted to number fields of degree less than 7. In [5], we compute low zeros by using Weil's explicit formula. More precisely, we reverse the construction that was used by Serre and Odlyzko to bound discriminants, in order to use it when the discriminant of the field is known. Therefore, we estimate the first zero of the Dedekind zeta function of totally imaginary fields of degree less than 30 having the smallest known discriminant. Unfortunately, this method is not efficient for computing high zeros, and in any case requires the computation of the norms of many prime ideals. It is therefore easier to compute the norms of prime ideals by using efficient algorithms for factoring the minimal polynomial modulo rational primes, and to obtain the contribution of the zeros by substitution in Weil's explicit formula. Hence, one relevant question, asked by Odlyzko in [4], concerns the relative contributions of prime ideals and zeros to the explicit formula for minimal discriminants. In the following section, Weil's explicit formula is stated, as is a lower bound for the discriminant that is derived from this formula, which forms the basis of the numerical computations.

2. *Weil's identity*

Consider functions $F : \mathbb{R} \to \mathbb{R}$, which are even and satisfy the following conditions.

(A) F is continuous and continuously differentiable everywhere except at a finite number of points a_i, where $F(x)$ and $F'(x)$ have only a discontinuity of the first kind, such that $F(a_i) = \frac{1}{2}(F(a_i + 0) + F(a_i - 0))$.

(B) There is a number $b > 0$ such that $F(x)$ and $F'(x)$ are $O(e^{-(1/2+b)|x|})$ as $|x| \to \infty$. Then the Mellin transform of F:

$$\Phi(s) = \int_{-\infty}^{+\infty} F(x) e^{(s-1/2)x} dx$$

is holomorphic in every strip $-a \leqslant \sigma \leqslant 1 + a$ where $0 < a < b$ and $a < 1$, and we have the following result, established by Weil (see [7] and [8]).

Theorem (Weil). *Let F satisfy the conditions* (A) *and* (B) *above. Then the sum $\sum \Phi(\rho)$ running over the non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta_K(s)$ with $|\gamma| < T$ tends to a limit as T tends to infinity, and this limit is given by the formula:*

$$\sum_{\rho} \Phi(\rho) = \Phi(0) + \Phi(1) - 2 \sum_{\mathfrak{p},m} \frac{\ln(N(\mathfrak{p}))}{N(\mathfrak{p})^{m/2}} F(m \ln(N(\mathfrak{p}))) + F(0) \left[\ln(|d_K|) - n \ln(2\pi)\right] - r_1 J(F) - n I(F), \tag{4}$$

where

$$J(F) = \int_0^{+\infty} \frac{F(x)}{2\,\mathrm{ch}(x/2)} dx \quad \textit{and} \quad I(F) = \int_0^{+\infty} \left(\frac{F(x)}{2\,\mathrm{sh}(x/2)} - \frac{e^{-x}}{x} \right) dx.$$

Consequences. If we consider positive functions F depending on a positive parameter y, such that $F(0) = 1$ and for which the Fourier transform is also positive, we get the inequality:

$$\ln(|d_K|) \geqslant -\Phi(0) - \Phi(1) + n(\ln(2\pi) + I(F)) + r_1 J(F). \tag{5}$$

In the case where F is either Serre's or Odlyzko's function, we determine the optimal value of y that gives the best lower bound for $\ln(|d_K|)$, obtaining inequalities (2) and (3).

3. *Key to Table 1*

The values given in Table 1 are presented in the following way. We denote by n the degree of an imaginary field of small discriminant, as given in [1]. We compute the optimal values of y that give the GRH Odlyzko bounds B_o and the Serre bounds B_s, respectively. Note that these bounds are slightly better than the published ones. We also compute the percentage E_o of the root discriminant $|d_{K_{min}}|^{1/n}$ above $\exp(B_o/n)$, as well as v_o, the contribution of the ideal primes, and E'_o, the percentage above $\exp((B_o + v_o)/n)$. Similarly, we compute the percentage E_s of the root discriminant above $\exp(B_s/n)$, the contribution v_s of the ideal primes, and the percentage E'_s above $\exp((B_s + v_s)/n)$.

4. *Comments*

The computational results in Table 1 indicate that the contribution of the zeros to the explicit formulae for minimal discriminants is larger than the contribution of the prime ideals, but that both of them seem to be of comparable magnitude. Evidently, by selecting Odlyzko's function, we obtain GRH bounds that are much closer to the discriminants of existing fields than with Serre's function. However, the contribution of the zeros to the latter function is smaller than in the case of a function with bounded support, such as that of Odlyzko. Actually Serre's function concentrates the sum over the zeros only on the very low zeros, which is roughly the main idea used for computing low zeros in [5], and the order n_χ of a zero of an Artin L-function at the central point s is equal to $\frac{1}{2}$ (see [6]).

Table 1: The contribution of zeros to the Odlyzko–Serre bounds

n = 8					
y_o	=	0.09576	y_s	=	0.478
B_o	=	13.9766614	B_s	=	13.79148
E_o	=	0.855589%	E_s	=	3.217392%
v_o	=	0.0211467	v_s	=	0.114526
E'_o	=	0.589344%	E'_s	=	1.750274%
n = 10					
y_o	=	0.07544	y_s	=	0.379
B_o	=	19.0660226	B_s	=	18.81308
E_o	=	0.939466%	E_s	=	3.525133%
v_o	=	0.0201688	v_s	=	0.151892
E'_o	=	0.736087%	E'_s	=	1.9645%
n = 12					
y_o	=	0.06308	y_s	=	0.319
B_o	=	24.3415208	B_s	=	24.01669
E_o	=	0.843225%	E_s	=	3.610240%
v_o	=	0.0115306	v_s	=	0.180580
E'_o	=	0.746373%	E'_s	=	2.0627%
n = 14					
y_o	=	0.05477	y_s	=	0.277
B_o	=	29.7576430	B_s	=	29.35786
E_o	=	0.581082%	E_s	=	3.494613%
v_o	=	0.0180307	v_s	=	0.212350
E'_o	=	0.451625%	E'_s	=	1.9366%
n = 16					
y_o	=	0.04879	y_s	=	0.248
B_o	=	35.2848987	B_s	=	34.80769
E_o	=	1.163917%	E_s	=	4.226583%
v_o	=	0.0468898	v_s	=	0.281873
E'_o	=	0.867879%	E'_s	=	2.4064%
n = 18					
y_o	=	0.04427	y_s	=	0.225
B_o	=	40.9029186	B_s	=	40.34632
E_o	=	1.377733%	E_s	=	4.561493%
v_o	=	0.0669201	v_s	=	0.347252
E'_o	=	0.981985%	E'_s	=	2.5636%
n = 20					
y_o	=	0.04073	y_s	=	0.208
B_o	=	46.5969446	B_s	=	45.95933
E_o	=	1.572763%	E_s	=	4.863109%
v_o	=	0.1166656	v_s	=	0.491621
E'_o	=	0.981985%	E'_s	=	2.3168%

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Table 1, *continued*

n = 22					
y_o	=	0.03787	y_s	=	0.194
B_o	=	52.3558796	B_s	=	51.63586
E_o	=	1.853671%	E_s	=	5.242294%
v_o	=	0.2399226	v_s	=	0.525236
E'_o	=	0.748933%	E'_s	=	2.7759%
n = 24					
y_o	=	0.03551	y_s	=	0.182
B_o	=	58.1711296	B_s	=	57.367554
E_o	=	1.348998%	E_s	=	4.799842%
v_o	=	0.0784337	v_s	=	0.468384
E'_o	=	1.018323%	E'_s	=	2.7743%
n = 26					
y_o	=	0.03352	y_s	=	0.172
B_o	=	64.0358757	B_s	=	63.147700
E_o	=	5.788320%	E_s	=	9.464547%
v_o	=	0.6124699	v_s	=	0.948136
E'_o	=	3.325436%	E'_s	=	5.5446%
n = 28					
y_o	=	0.03183	y_s	=	0.164
B_o	=	69.9446011	B_s	=	68.970908
E_o	=	1.135052%	E_s	=	4.713863%
v_o	=	0.0904462	v_s	=	0.541031
E'_o	=	0.808890%	E'_s	=	2.7099%
n = 30					
y_o	=	0.03036	y_s	=	0.156
B_o	=	75.8927614	B_s	=	74.832739
E_o	=	1.720601%	E_s	=	5.379056%
v_o	=	0.0865725	v_s	=	0.635029
E'_o	=	1.427484%	E'_s	=	3.1718%
n = 32					
y_o	=	0.02908	y_s	=	0.150
B_o	=	81.8765639	B_s	=	80.729609
E_o	=	1.113499%	E_s	=	4.825654%
v_o	=	0.1139595	v_s	=	0.634821
E'_o	=	0.775468%	E'_s	=	2.7665%
n = 36					
y_o	=	0.02694	y_s	=	0.139
B_o	=	93.9387309	B_s	=	92.615850
E_o	=	1.709112%	E_s	=	5.516104%
v_o	=	0.1994260	v_s	=	0.812826
E'_o	=	1.147241%	E'_s	=	3.1604%

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Table 1, *continued*

n = 40					
y_o	=	0.02521	y_s	=	0.131
B_o	=	106.110512	B_s	=	104.609693
E_o	=	1.543488%	E_s	=	5.425825%
v_o	=	0.2045192	v_s	=	0.867687
E'_o	=	1.025623%	E'_s	=	3.1635%
n = 48					
y_o	=	0.02260	y_s	=	0.118
B_o	=	130.724149	B_s	=	128.862258
E_o	=	1.005948%	E_s	=	5.000888%
v_o	=	0.1173174	v_s	=	0.942677
E'_o	=	0.759379%	E'_s	=	2.9588%

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