London Mathematical Society ISSN 1461–1570

THE SOLVABLE PRIMITIVE PERMUTATION GROUPS OF DEGREE AT MOST 6560

B. EICK AND B. HÖFLING

Abstract

The authors present an algorithm to construct conjugacy class representatives of the solvable primitive subgroups of S_d for a given degree d. Using this method, they determine the solvable primitive permutation groups of degree at most 6560 (that is, $3^8 - 1$), up to conjugacy.

1. *Introduction*

Primitive permutation groups play a role in various areas of group theory. For example, the maximal subgroups of a finite group correspond to its primitive permutation actions. Further, the primitive permutation groups can be considered as the building blocks of all permutation groups. Consequently, primitive permutation groups have received a considerable amount of attention.

The primitive permutation groups are classified by the O'Nan–Scott Theorem [**14**]. They divide naturally into two types: those with solvable socle, called the affine groups, and those with insolvable socle. The primitive permutation groups with insolvable socle and degree at most 1000 have been determined up to conjugacy by Dixon and Mortimer [**2**].

If the socle of a primitive permutation group G is solvable, then the degree of G is a prime power, say p^n, and $S = \mathrm{Soc}(G)$ is elementary abelian of order p^n. Further, G splits over S and the action of G/S on S induces an embedding of G/S into $\mathrm{GL}(n, p)$. This yields a well-known one-to-one correspondence between the conjugacy classes of affine primitive subgroups of S_{p^n} and the conjugacy classes of irreducible subgroups of $\mathrm{GL}(n, p)$.

The irreducible subgroups of $\mathrm{GL}(n, p)$ divide naturally into two kinds: the solvable and the insolvable ones. There are many results known on the insolvable irreducible subgroups of $\mathrm{GL}(n, p)$. Systematic attempts to determine them up to conjugacy for $p^n \leqslant 1000$ are due to Theißen [**24**] and Roney-Dougal and Unger [**17**].

The conjugacy classes of solvable irreducible subgroups of $\mathrm{GL}(n, p)$ for $p^n \leqslant 255$ have been determined by Short [**18**], using a variety of theoretical results together with algorithmic approaches for this purpose. Recently, Hulpke observed that two conjugacy classes of groups are missing in Short's library of groups. This has prompted us to consider new approaches for an algorithmic determination of these groups.

As a result, we present an effective algorithm to construct the solvable irreducible subgroups of $\mathrm{GL}(n, p^l)$ up to conjugacy. As noted above, for $l = 1$ these groups correspond to the conjugacy classes of solvable primitive subgroups of S_{p^n}. Using our algorithm, we have determined them for $p^n \leqslant 6560$. Thus we have obtained a corrected version of Short's group library, and have significantly extended the previously known classifications.

Received 13 February 2002, revised 14 October 2002; *published* 25 February 2003.
2000 Mathematics Subject Classification 20-04, 20B15, 20D10, 20H30

---

(1) Determine a list $\mathcal{M}$ of solvable irreducible subgroups of $\mathrm{GL}(n, q)$ such that $\mathcal{M}$ contains a conjugate of each maximal solvable irreducible subgroup of $\mathrm{GL}(n, q)$.

(2) Initialize $\mathcal{S} = \emptyset$.

(3) For each M in $\mathcal{M}$ do:

   (a) initialize $\mathcal{T} = \{M\}$;

   (b) while $\mathcal{T} \neq \emptyset$ do:

      (i) choose $G \in \mathcal{T}$ and delete G from $\mathcal{T}$;

      (ii) check if G is conjugate to a group in $\mathcal{S}$ under $\mathrm{GL}(n, p)$;

      (iii) if this is not the case, then:

      - add G to $\mathcal{S}$;
      - compute the maximal subgroups $\mathcal{U}$ in G up to conjugacy in $N_M(G)$;
      - delete the reducible subgroups from $\mathcal{U}$;
      - append the remaining subgroups in $\mathcal{U}$ to $\mathcal{T}$.

(4) Return the resulting list $\mathcal{S}$.

Algorithm 1: SolvableIrreducibleGroups(n, q)

2. *Construction of solvable irreducible matrix groups*

In this section we describe the basic outline of the approach that we have used to construct – up to conjugacy – the solvable irreducible subgroups of $\mathrm{GL}(n, q)$ for $q = p^l$ a prime power. We first give a top-level outline of this method in Algorithm 1, and then we discuss details of the method below.

Our method for Step (1) is based on Aschbacher's classification of the maximal subgroups of $\mathrm{GL}(n, q)$. Section 4 contains an outline of this method. Step (3)(b)(ii) is facilitated by an effective conjugacy test for solvable irreducible subgroups in $\mathrm{GL}(n, q)$. Our algorithm for this purpose is introduced in Section 3. The remaining steps in the above algorithm are obtained by standard techniques in algorithmic group theory. We include a brief description of them in the remainder of this section, and we refer to Section 5 for further references and background material on these methods.

Each group $M \in \mathcal{M}$ considered in the above algorithm is finite and solvable. Thus there exists an isomorphism $\phi_M : M \longrightarrow \overline{M}$ from the matrix group M to a polycyclically presented group $\overline{M}$, which we compute using the methods described in [**19**] for a faithful permutation representation of M on a suitable subset of the underlying vector space. It is well known that polycyclic presentations facilitate effective computations with the groups they define; see, for example, [**13**] and [**20**]. We determine an isomorphism ϕ_M for each M within the construction of M as outlined in Section 4.

To compute the maximal subgroups of $G^{\phi_M} = \overline{G} \leqslant \overline{M}$ up to conjugacy in $N_{\overline{M}}(\overline{G})$, we use a variation of the methods described in [**11**] for an effective iterated computation of conjugacy classes of subgroups and their normalizers. For each computed subgroup $\overline{U} \leqslant \overline{G}$, we determine its preimage U under ϕ_M. Using the MeatAxe algorithm (see [**8**] and [**12**]), we can then check whether U acts reducibly.

In summary, these approaches yield effective realizations of the steps in (3)(b)(iii) of the algorithm SolvableIrreducibleGroups (Algorithm 1). Further details on their implementation and their performance are given in Section 5.

---

(1) We compute a set of invariants for G_1 and G_2 which are preserved by conjugacy in $\mathrm{GL}(n, q)$.
If these invariants differ in G_1 and G_2, then the groups cannot be conjugate.

(2) We check whether a conjugating element in $\mathrm{GL}(n, q)$ exists for G_1 and G_2.
If so, then we determine such an element explicitly.

Algorithm 2: ConjugacyMatrixGroups(G_1, G_2)

3. *Testing the conjugacy of solvable irreducible matrix groups*

We present a method to check whether two irreducible subgroups G_1 and G_2 of $\mathrm{GL}(n, q)$ are conjugate in $\mathrm{GL}(n, q)$. While our method can be applied to irreducible subgroups in general, it will be particularly effective if the groups being considered are solvable, and if embeddings into polycyclically presented groups are available for them.

We give a top-level outline of our approach in Algorithm 2, and we discuss details of this algorithm below. We note that this general approach is valid for all subgroups of $\mathrm{GL}(n, p)$. The irreducibility will be used in an effective method for Step (2) of the algorithm.

The invariants used in Step (1) are based on the conjugacy classes of elements of the groups under consideration: we determine these conjugacy classes, and for each class we use its length as well as the order and the characteristic polynomial of a representative of this class for the invariants. If the groups are solvable, and embeddings into polycyclically presented groups are given, then the determination of the conjugacy classes of elements can be performed effectively in the polycyclically presented image; see [**15**].

Next, we consider Step (2) in Algorithm 2 (ConjugacyMatrixGroups). Clearly, if a conjugating element for G_1 and G_2 exists, then G_1 and G_2 are isomorphic. Conversely, we call an isomorphism from G_1 to G_2 *linear* if it is induced by conjugation with an element from $\mathrm{GL}(n, q)$. We use the following lemma to check the linearity of a given isomorphism between two irreducible subgroups G_1 and G_2.

LEMMA 3.1. *Consider $G_1, G_2 \leqslant \mathrm{GL}(n, q)$ with an isomorphism $\alpha : G_1 \longrightarrow G_2$. Let $G_1 = \langle g_1, \ldots, g_d \rangle$, and write $f_i = g_i^\alpha$ for $1 \leqslant i \leqslant d$. Denote*

$$C = \{c \in M(n, q) \mid g_i c - c f_i = 0, \text{ for } 1 \leqslant i \leqslant d\},$$

where $M(n, q)$ is the full matrix ring.

(a) *α is linear if and only if there exists an invertible element $c \in C \setminus \{0\}$.*

(b) *If G_1 is irreducible, then either all or none of the elements of $C \setminus \{0\}$ are invertible.*

Proof. (a) If α is linear, then it induces a conjugating element $c \in C$ with $c \neq 0$. Clearly, c is invertible. The converse is also obvious.

(b) Suppose that there exists an invertible element $c \in C \setminus \{0\}$. Then $C = Ec$, where $E = \{e \in M(n, q) \mid g_i e - e g_i = 0 \text{ for } 1 \leqslant i \leqslant d\}$ is the endomorphism ring of G_1. Since G_1 is irreducible, its endomorphism ring is a finite skew-field, and thus a field. Hence all elements of $C \setminus \{0\}$ are invertible. □

Lemma 3.1 yields an effective method of testing whether a given isomorphism α is linear. First, we determine a small generating set $g_1, \ldots, g_d$ for G_1 and its images $f_1, \ldots, f_d$. Next,

(1) Compute an explicit isomorphism $\beta : G_1 \longrightarrow G_2$.
If no such isomorphism exists, then G_1 and G_2 cannot be conjugate, and we return 'fail'.
(2) Compute a small generating set $g_1, \ldots, g_d$ for G_1.
(3) Compute $\mathrm{Aut}(G_1)$.
(4) Enumerate a transversal T for $\mathrm{Lin}(G_1) \setminus \mathrm{Aut}(G_1)$:
   (a) check for each $\gamma \in T$ whether $\gamma\beta$ is linear;
   (b) if so, then return its corresponding conjugating element c.
(5) G_1 and G_2 are not conjugate, and thus we return 'fail'.

Algorithm 3: ConjugacyIrreducibleMatrixGroups(G_1, G_2)

it is straightforward to compute C as the set of solutions to the linear equations $g_i c - c f_i = 0$ for $1 \leqslant i \leqslant d$. Once C is given, it remains to choose an arbitrary element $c \in C \setminus \{0\}$, if possible, and to check whether c is invertible.

A connection between the automorphisms of G_1 and the isomorphisms $G_1 \longrightarrow G_2$ is summarized in the following obvious lemma.

LEMMA 3.2. *Let $G_1, G_2 \leqslant \mathrm{GL}(n, q)$, and let $\beta : G_1 \longrightarrow G_2$ be an isomorphism.*
(a) *Each isomorphism $\alpha : G_1 \longrightarrow G_2$ is of the form $\alpha = \gamma\beta$ for some $\gamma \in \mathrm{Aut}(G_1)$.*
(b) *Let α_1 and α_2 be two isomorphisms between G_1 and G_2 such that $\alpha_i = \gamma_i\beta$ and $\gamma_1 = \delta\gamma_2$ for some linear automorphism $\delta \in \mathrm{Aut}(G_1)$. Then α_1 is linear if and only if α_2 is linear.*

As described in Lemma 3.2(a), we compute the set of all isomorphisms between G_1 and G_2 as $\{\alpha\beta \mid \alpha \in \mathrm{Aut}(G_1)\}$ for a fixed isomorphism $\beta : G_1 \longrightarrow G_2$. We denote

$$\mathrm{Lin}(G_1) = \{\alpha \in \mathrm{Aut}(G_1) \mid \alpha \text{ is linear}\},$$

and we consider a transversal T for $\mathrm{Lin}(G_1) \setminus \mathrm{Aut}(G_1)$. Then, by Lemma 3.2(b), it is sufficient to search the set $T\beta$ to find a linear isomorphism between G_1 and G_2.

Note that a transversal T can be enumerated via the orbit of $\mathrm{Aut}(G_1)$ acting by right multiplication on the cosets for $\mathrm{Lin}(G_1) \setminus \mathrm{Aut}(G_1)$. The linearity check of Lemma 3.1 facilitates an efficient method for testing whether two elements of $\mathrm{Aut}(G_1)$ are contained in the same coset of $\mathrm{Lin}(G_1)$.

A summary of our resulting method of computing a conjugating element between two irreducible subgroups $G_1, G_2 \leqslant \mathrm{GL}(n, q)$ is given in Algorithm 3

If G_1 and G_2 are both solvable, and if embeddings into polycyclically presented groups are available for them, then constructive isomorphism tests have been described in [**10**] and, in the p-group case, in [**16**]. The former is based upon ideas from [**5**] and [**7**]. Generators for $\mathrm{Aut}(G_1)$ can be obtained using a polycyclically presented image of G_1 as described in [**21**] and [**4**].

Finally, we note that a brute-force backtracking search for a conjugating element in $\mathrm{GL}(n, q)$ is usually less effective than the method proposed in this section.

4. *Maximal solvable irreducible matrix groups*

Aschbacher [1] introduced a classification for the subgroups of $\mathrm{GL}(n,q)$. Using the notation of [9], Aschbacher's classification sorts these subgroups into nine classes; we include a brief overview of these classes below.

THEOREM 4.1 (ASCHBACHER). *Let $q = p^l$, and let $V = \mathbb{F}_q^n$. Consider $G \leqslant \mathrm{GL}(n,q)$, and denote $Z = Z(\mathrm{GL}(n,q))$. Then one of the following holds.*

(1) *G acts reducibly on V.*

(2) *G acts imprimitively on V.*

(3) *G preserves a tensor decomposition of V.*

(4) *G preserves a symmetric tensor power decomposition of V.*

(5) *A conjugate of G embeds into $\Gamma L(n/m, q^m)$, the group of semilinear maps.*

(6) *A conjugate of G embeds into $\mathrm{GL}(n, p^e)Z$ with $e \mid l$.*

(7) *G normalizes an irreducible extraspecial or symplectic-type group.*

(8) *$H' \leqslant G \leqslant HZ$ for a classical group H.*

(9) *$G/(G \cap Z)$ is almost non-abelian simple.*

Our aim is to determine a list of solvable irreducible subgroups of $\mathrm{GL}(n,q)$ that contains at least one conjugate of each maximal solvable irreducible subgroup. This is a trivial task if $\mathrm{GL}(n,q)$ is itself solvable.

REMARK 4.2. The group $\mathrm{GL}(n,q)$ is solvable if and only if $n = 1$ or $(n,q) \in \{(2,2),(2,3)\}$.

In the discussion that follows, we assume that n and q are chosen such that $\mathrm{GL}(n,q)$ is insolvable. Then we consider each of the Aschbacher classes in turn, and we construct up to conjugacy the maximal solvable irreducible subgroups G in this class. Obviously, we want to avoid Class (1), and Class (9) cannot yield a solvable group. The other classes are considered here. We denote $Z = Z(\mathrm{GL}(n,q))$, and we note that $Z \subseteq G$.

Class (2). Let G be imprimitive, and let $V = V_1 \oplus \ldots \oplus V_r$ be a minimal system of imprimitivity. Then $n = rm$, and the subspaces $V_1, \ldots, V_r$ are permuted transitively by G. Thus G is a wreath product $H \wr K$, where H is a maximal solvable primitive subgroup of $\mathrm{GL}(m,q)$ and K is a maximal solvable transitive subgroup of S_r.

Class (3). Suppose that G preserves a tensor decomposition, and let $V = V_1 \otimes \ldots \otimes V_r$ be a minimal G-invariant tensor decomposition. Then $n = m_1 \cdots m_r$ for $m_i = \dim(V_i)$ and G/Z is a direct product $G_1 \times \ldots \times G_r$ for r maximal solvable primitive tensor-indecomposable subgroups $G_i \leqslant \mathrm{PGL}(m_i,q)$; also, G can be constructed using iterated Kronecker products.

Class (4). Suppose that G preserves a symmetric tensor power, and let $V = V_1 \otimes \ldots \otimes V_r$ be a minimal G-invariant symmetric tensor power. Then $n = m^r$, and G permutes the components $V_1, \ldots, V_r$. Thus G/Z is a wreath product $H \wr K$, where H is a maximal solvable primitive tensor-indecomposable subgroup of $\mathrm{PGL}(m,q)$ and K is a maximal solvable transitive subgroup of S_r.

Class (5). Suppose that G embeds into $\Gamma L(n/m, q^m)$. Then G contains a maximal solvable irreducible subgroup H of $\mathrm{GL}(n/m, q^m)$ as a normal subgroup. Since $N_{\mathrm{GL}(n,q)}(H) = H$, we obtain $G = N_{\Gamma L(n/m,q^m)}(H)$.

Class (6). Suppose that G embeds into $\mathrm{GL}(n, p^e)Z$ with $e \mid l$. Then $G = HZ$ for a maximal solvable irreducible subgroup H of $\mathrm{GL}(n, p^e)$.

Class (7). Assume that G normalizes an irreducible extraspecial or symplectic-type group E. Then $n = r^m$ for a prime r, and r divides $q - 1$. The group E is an extraspecial r-group of order r^{2m+1} or $r = 2$, and E is a symplectic type-2 group of order 2^{2m+2}. For each such E, there exists a unique embedding of E as subgroup in $\mathrm{GL}(n, q)$ up to conjugacy; see Remark 4.3 below. We obtain $C_{\mathrm{GL}(n,q)}(E) = Z$, and we determine $N = N_{\mathrm{GL}(n,q)}(E)$ as extension of Z by the group of linear automorphisms $\mathrm{Lin}(E)$ of Section 3. Then G/E is a maximal solvable subgroup of N/E. These can be computed using the subgroup lattice of N/E if N/E is insolvable.

Class (8). The classical subgroups of $\mathrm{GL}(n, q)$ are investigated in [**23**]. For most such subgroups H, the derived subgroup H' is insolvable. The exceptions are (if the trivial cases $n = 1$ or $n = 2$ and $q \in \{2, 3\}$ are excluded):

- unitary groups: $U(2, 2^2)$, $U(2, 3^2)$ and $U(3, 2^2)$;
- orthogonal groups: $O^{\pm}(2, q)$, $O(3, 2)$, $O(3, 3)$, $O^+(4, 2)$ and $O^+(4, 3)$.

In each of these cases we find that the groups under consideration arise in one of the other Aschbacher cases:

- $U(2, 2^2)$ is imprimitive;
- $U(2, 3^2)$ and $U(3, 2^2)$ normalize a symplectic group of order 16 or an extraspecial group of order 27;
- $O^+(2, q)$ is dihedral and imprimitive and $O^-(2, q)$ is semilinear;
- $O(3, 2)$ is reducible, and $O(3, 3)$ is imprimitive;
- $O^+(4, 2)$ is imprimitive, and $O^+(4, 3)$ normalizes an extraspecial group of order 32.

In summary, we can ignore Class (8) for our purposes.

For Class (7) we add the following remark.

Remark 4.3. Let $n = r^m$ for a prime r, and let $q = p^l$ a prime power with $r \mid q - 1$.

(a) Let E be an extraspecial r-group of order r^{2m+1}. Then there exists exactly one conjugacy class of subgroups of $\mathrm{GL}(n, q)$ isomorphic to E. More precisely, E has $r - 1$ modules of dimension n over $\mathbb{F}_q$ as described in [**3**, B, Theorems 9.16 and 9.17]. These lead to conjugate subgroups by [**3**, A, Theorem 20.8].

(b) Let E be a symplectic type 2-group of order 2^{2m+2}. Then E is a central product of an extraspecial 2-group F of order 2^{2m+1} and a cyclic group C of order 4. If $E \leqslant \mathrm{GL}(n, q)$, then, by Schur's lemma, $C \leqslant Z$ and F is irreducible. Thus we find that there exists exactly one conjugacy class of subgroups of $\mathrm{GL}(n, q)$ isomorphic to E, by (a).

The above constructions make use of the solvable irreducible subgroups of $\mathrm{GL}(m, p^e)$ for a smaller dimension m or a smaller field p^e. We assume that these are known by induction. Further, we need the maximal solvable transitive subgroups of S_d for small degrees d. These can be constructed as wreath products $P \wr T$, where P is a maximal solvable primitive subgroup of S_{r^k} for a prime r, and T is a maximal solvable transitive subgroup of S_{d/r^k}.

REMARK 4.4. (a) The Aschbacher classes of subgroups of $\mathrm{GL}(n, q)$ are not necessarily disjoint. Thus we may obtain conjugate subgroups in two different classes, or we may encounter solvable irreducible subgroups that are non-maximal, with this property. Note that this does not affect our requirements.

(b) In each case we determine the relevant groups as subgroups of $\mathrm{GL}(n, q)$. In later applications, we also need a polycyclic presentation for these groups. In most cases, this can readily be obtained from the construction of the group; for example, it is straightforward to construct polycyclic presentations for direct products (Class (3)), wreath products (Classes (2) and (4)), and extensions (Class (5)). The only case that is not obvious in this respect is Class (7). Since the groups arising in this case are generally of small order, we can construct polycyclic presentations for them by brute force.

Short [**18**] and Suprunenko [**22**] include a further structural analysis of the maximal solvable primitive subgroups of $\mathrm{GL}(n, q)$.

5. *Implementation and results*

We have implemented the above algorithms in the computer algebra system GAP [**6**]. Using our implementation, we have determined the solvable irreducible subgroups of $\mathrm{GL}(n, p)$ for $p^n \leqslant 6560$. Our method clearly extends to other values of p and n as well. However, the case $p^n = 3^8 = 6561$ seems to be too hard for our implementation. In general, the computation of abstract isomorphisms in Algorithm 3 (ConjugacyIrreducibleMatrixGroups) of Section 3 is the main bottleneck in our method. Also, the method used for the computation of conjugacy classes of subgroups reaches its limits for some large maximal solvable subgroups of $\mathrm{GL}(n, p)$.

The resulting group library is intended for publication in GAP. In the following sections we include a report on this group library and its determination. We also outline further applications of our method, and we give some indications of its limits. Finally, we comment on the reliability of the computed data.

5.1. *The solvable irreducible groups for* $p^n \leqslant 6560$

Table 1 contains the numbers of conjugacy classes of the solvable irreducible subgroups of $\mathrm{GL}(n, p)$ for $p^n \leqslant 6560$. If $n = 1$, then $\mathrm{GL}(1, p) \cong C_{p-1}$, and each subgroup of $\mathrm{GL}(1, p)$ is solvable irreducible. Hence this case is trivial, and we consider only the case $n > 1$. Note that the numbers of groups that arise depend significantly on the prime divisors of n and $p - 1$.

It takes twelve minutes to determine the groups of Table 1 for $p^n \leqslant 255$, eighteen hours for $p^n \leqslant 4095$, and four days for $p^n = 2^{12} = 4096$ using a Pentium III PC under Linux. Most of the CPU time is spent on computing abstract isomorphisms in Algorithm 3 (ConjugacyIrreducibleMatrixGroups) of Section 3. We note that the invariant computation of Step (1) of Algorithm 2 (ConjugacyMatrixGroups) identifies all the non-conjugate groups for almost all values of n and p in the range under consideration.

5.2. *The case when n is a prime*

If the dimension n is a prime, then the method of Section 4 reduces significantly, and the maximal solvable irreducible subgroups of $\mathrm{GL}(n, q)$ can be readily determined. A classification of these groups is given in [**22**, p. 167].

Table 1: Numbers of the solvable irreducible subgroups of $\mathrm{GL}(n, p)$ for $p^n \leqslant 6560$.

n	2	3	4	5	6	7	8	9	10	11	12
$p = 2$	2	2	10	2	40	2	129	21	50	6	934
$p = 3$	7	9	108	16	324	18					
$p = 5$	19	22	509	48							
$p = 7$	29	62	894								
$p = 11$	42	54									
$p = 13$	62	136									
$p = 17$	75	66									
$p = 19$	77										
$p = 23$	54										
$p = 29$	100										
$p = 31$	114										
$p = 37$	127										
$p = 41$	174										
$p = 43$	118										
$p = 47$	66										
$p = 53$	100										
$p = 59$	82										
$p = 61$	212										
$p = 67$	118										
$p = 71$	192										
$p = 73$	261										
$p = 79$	166										

Table 2 gives an overview of their possible group orders and Aschbacher classes. In particular, there are at most six conjugacy classes of maximal solvable irreducible subgroups of $\mathrm{GL}(n, q)$ if n is prime, but not all of them occur in all the cases listed. More details are given in [22].

Table 2: The maximal solvable irreducible subgroups for primes n

Order	Necessary condition	Aschbacher class	
$n(n-1)(q-1)^n$		imprimitive	(2)
$n(q^n-1)$		semilinear	(5)
$2(n+1)n^2(q-1)$	$q \equiv 1 \bmod n$	extraspecial	(7)
$2(n-1)n^2(q-1)$	$q \equiv 1 \bmod n$	extraspecial	(7)
$24n^2(q-1)$	$q \equiv 1 \bmod n$ $n \equiv \pm 3 \bmod 8$	extraspecial	(7)
$48n^2(q-1)$	$q \equiv 1 \bmod n$, $n \equiv \pm 1 \bmod 8$	extraspecial	(7)
$48n^2(q-1)$	$q \equiv 1 \bmod n$, $n \equiv \pm 1 \bmod 8$	extraspecial	(7)

Table 3: Numbers of maximals and largest orders in $\mathrm{GL}(n, 2)$

n	p^n	$\lvert\mathcal{M}\rvert$	$\sharp$ maximals	largest order
2	4	1	1	6
3	8	1	1	21
4	16	4	2	72
5	32	1	1	155
6	64	8	4	1 296
7	128	1	1	889
8	256	16	3	31 104
9	512	4	2	55 566
10	1024	7	3	155 520
11	2048	1	1	22 517
12	4096	48	9	4 667 544
13	8192	1	1	106 483
14	16384	?	?	11 757 312
15	32768	?	?	81 682 020
16	65536	?	?	1 934 917 632

In summary, the case when n is a prime is comparatively easy to handle. Using our algorithm, we can determine the solvable irreducible subgroups of $\mathrm{GL}(2, p)$ for all $p \leqslant 100$ at least.

5.3. *The case* $p = 2$

In Table 3, we outline a list of orders of the number of maximal solvable irreducible subgroups, and the order of the largest maximal solvable irreducible subgroups in $\mathrm{GL}(n, 2)$. We also include the lengths of the lists $\mathcal{M}$ of candidates for the maximal solvable irreducible subgroups of $\mathrm{GL}(n, p)$, as determined in Section 4.

We can readily determine the solvable irreducible subgroups in $\mathrm{GL}(13, 2)$. In fact, this is easier than the corresponding computation in $\mathrm{GL}(12, 2)$. However, the algorithm that we used to compute the subgroups ran out of memory while computing the irreducible solvable subgroups of $\mathrm{GL}(14, 2)$.

5.4. *Maximal solvable irreducible subgroups in the remaining cases*

Finally, we list the largest orders of the maximal solvable irreducible subgroups in the remaining cases in Table 4. We include the case $p^n = 3^8$ to give an indication of why this case is harder than the smaller cases that we have dealt with.

5.5. *Comments on the reliability of the data*

The library of solvable irreducible groups of degree at most 6560 has been computed using our GAP implementation, without user interaction. However, the risk remains that there are mistakes in our own implementation, or mistakes in the GAP methods used. To minimize these risks, we have performed systematic cross-checks with existing data libraries.

Table 4: Largest orders of the maximal solvable irreducible subgroups of $\mathrm{GL}(n, p)$

p	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$
$p = 3$	78	4 608	1 210	663 552	15 302	127 401 984
$p = 5$	384	18 432	20 480			
$p = 7$	1 296	41 472				
$p = 11$	6 000					
$p = 13$	10 368					
$p = 17$	24 576					

First, we have compared our results with the group library computed by Short [**18**], which is available in GAP. Except for the two known omissions in Short's library, these two libraries agree with each other.

Secondly, we have systematically determined all the faithful irreducible representations of all the solvable groups of order at most 1000, except for 512 and 768. These groups are available in the small-groups library in GAP; the database presented in this paper will also be added to the GAP archive [**6**] in due course.

References

1. Michael Aschbacher, 'On the maximal subgroups of the finite classical groups', *Invent. Math.* 76 (1984) 469–514. 33

2. John D. Dixon and Brian Mortimer, 'The primitive permutation groups of degree less than 1000', *Math. Proc. Cambridge Philos. Soc.* 103 (1988) 213–238. 29

3. Klaus Doerk and Trevor Hawkes, *Finite solvable groups* (De Gruyter, Berlin/New York, 1992). 34, 34

4. Bettina Eick, C. R. Leedham-Green and E. A. O'Brien, 'Constructing automorphism groups of p-groups', *Comm. Algebra* 30 (2002) 2271–2295. 32

5. Volkmar Felsch and Joachim Neubüser, 'On a programme for the determination of the automorphism group of a finite group', *Computational Problems in Abstract Algebra* (*Proc. Conf., Oxford, 1967*) (Pergamon, Oxford, 1970) 59–60. 32

6. The GAP Group, GAP– *Groups, Algorithms, and Programming*, Version 4.3 (The GAP Group, Aachen/St Andrews, 2002); `http://www.gap-system.org`. 35, 38

7. D. F. Holt and Sarah Rees, 'Testing for isomorphism between finitely presented groups', *Groups, combinatorics & geometry* (*Durham, 1990*), London Math. Soc. Lecture Note Ser. 165 (Cambridge Univ. Press, Cambridge, 1992) 459–475. 32

8. Derek F. Holt and Sarah Rees, 'Testing modules for irreducibility', *J. Austral. Math. Soc. Ser. A* 57 (1994) 1–16. 30

9. Derek F. Holt, Charles R. Leedham-Green, Eamonn A. O'Brien and Sarah Rees, 'Computing matrix group decompositions with respect to a normal subgroup', *J. Algebra* 184 (1996) 818–838. 33

10. Alexander Hulpke, 'Konstruktion transitiver Permutationsgruppen', Dissertation, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, 1996. 32

11. Alexander Hulpke, ‘Computing subgroups invariant under a set of automorphisms’, *J. Symb. Comput.* 27 (1999) 415–427. 30

12. Gábor Ivanyos and Klaus Lux, ‘Treating the exceptional cases of the MeatAxe’, *Experiment. Math.* 9 (2000) 373–381. 30

13. Reinhard Laue, Joachim Neubüser and Ulrich Schoenwaelder, ‘Algorithms for finite soluble groups and the SOGOS system’, *Computational Group Theory, Proceedings LMS Symposium on Computational Group Theory, Durham 1982* (ed. Michael D. Atkinson, Academic Press, London, 1984) 105–135. 30

14. Martin W. Liebeck, Cheryl E. Praeger and Jan Saxl, ‘On the O’Nan–Scott theorem for finite primitive permutation groups’, *J. Austral. Math. Soc. Ser. A* 44 (1988) 389–396. 29

15. Matthias Mecky and Joachim Neubüser, ‘Some remarks on the computation of conjugacy classes of soluble groups’, *Bull. Austral. Math. Soc.* 40 (1989) 281–292. 31

16. E. A. O’Brien, ‘Isomorphism testing for p-groups’, *J. Symbolic Comput.* 17 (1) (1994) 133–147. 32

17. Colva M. Roney-Dougal and William R. Unger, ‘The affine primitive permutation groups of degree less than 1000’, *J. Symbolic Comput.*, to appear. 29

18. Mark W. Short, *The primitive soluble permutation groups of degree less than* 256, Lecture Notes in Math. 1519 (Springer, Berlin/Heidelberg, 1992). 29, 35, 38

19. Charles C. Sims, ‘Computing the order of a solvable permutation group’, *J. Symbolic Comput.* 9 (1990) 699–705. 30

20. Charles C. Sims, *Computation with finitely presented groups* (Cambridge Univ. Press, Cambridge, 1994). 30

21. Michael J. Smith, ‘Computing automorphisms of finite soluble groups’, PhD thesis, Australian National University, Canberra, Australia, 1994. 32

22. Dimitriĭ A. Suprunenko, *Matrix groups*, Transl. Math. Monogr. 45 (Amer. Math. Soc., Providence, RI, 1976). 35, 35, 36

23. Donald E. Taylor, *The geometry of the classical groups* (Heldermann Verlag, Berlin, 1992). 34

24. Heiko Theissen, ‘Eine Methode zur Normalisatorberechnung in Permutationsgruppen mit Anwendungen in der Konstruktion primitiver Gruppen’, Dissertation, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, 1997. 29

B. Eick `beick@tu-bs.de`
`http://www.tu-bs.de/~beick`
B. Höfling `b.hoefling@tu-bs.de`
`http://www.tu-bs.de/~bhoeflin`

Institut für Geometrie
Technische Universität
Pockelsstr. 14
38106 Braunschweig
Germany